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Geometric Model for Fundamental Particles

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An attempt is made to show that fundamental particles are manifestations of the geometry of space-time. This is done by demonstrating the existence of a purely geometrical model, which we have called *spherical rotation*, that satisfies Dirac's equation. The model is developed and illustrated both mathematically and mechanically. It indicates that the mass of a particle is entirely due to the spinning of the space-time continuum. Using the model, we can show the distinction between spin-up and spin-down states and also between particle and antiparticle states. It satisfies Einstein's criteria for a model that has both wave and particle properties, and it does so without introducing a singularity into the continuum.

1. INTRODUCTION

The oldest method of recording motion is quite simply the method of laying down a trail. The trail connects the start to the end of the journey. Take the example of a dog tied by a long rope to a tree. He runs twice around his kennel and then lies down. His master returns, sees the rope, and instantly knows that the dog has circled his house twice.

In any theory of the continuum that purports to describe matter as a distortion of space in the manner first suggested by W. K. Clifford,[2] the motions of the matter must not destroy the continuity of the space. Any set of curvilinear coordinates used to map the space in the vicinity of the particle must, like the rope attached to the dog, participate in the allowed motions of the space. This requirement, we have been able to show (see Appendix), is equivalent to the assertion that the only allowable motions must be represented by a *simply connected group*, namely, the *universal covering group* of the Lie group used to describe a local part of the motion.

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[2]W. K. Clifford (1956). All those who searched for a unified field theory subscribed to a geometrical interpretation of matter, including Einstein, Weyl, and Eddington.

This model shows that a persistent particle must be described by a *compact group*. The simplest compact universal covering group is named $SU(2)$. The model to be described has $SU(2)$ as its motion group.

2. THE MECHANICAL MODEL: THE SPHERICAL ROTATOR

The mechanical model that led to the geometric solution of the Dirac equation can be constructed roughly by suspending a practice golf ball from flexible wires that are, in turn, fixed onto a wooden framework. It is sufficient to choose six wires corresponding to the positive and negative axes of a three-dimensional coordinate system (Figure 1). The ball can always be rotated indefinitely without the wires becoming entangled. Each double revolution returns the system to its original configuration, provided each wire is kept in the correct relation to all the others.

The simplest workable relation is one that requires all spheres concentric with the ball to remain rigid; that is to say, the six points where the wires intersect an imagined spherical shell surrounding the ball must remain equidistant from one another, though the shell as a whole is free to turn about the center.

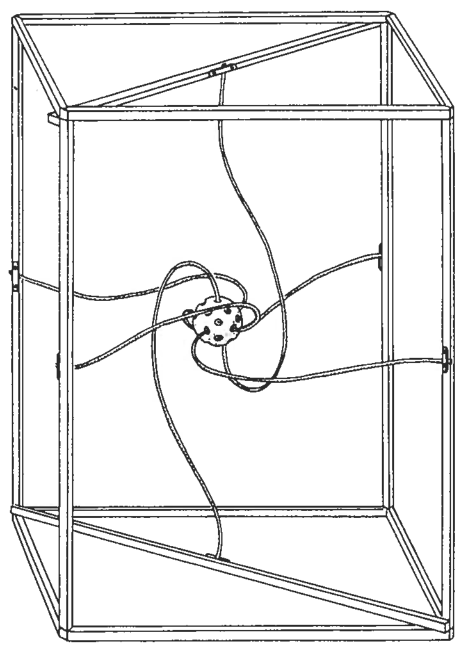

Fig. 1. Mechanical model for demonstrating spherical rotation.

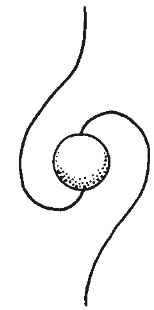

Fig. 2. Configuration of the z axis of the model.

Whereas the ball of this model can, in principle, be turned about any axis starting from any initial position, in practice, because of the restraints (flexibility, extensibility, etc.) of the materials used, there is a best way for demonstrating the motion:

Position the ball so that the point where each wire is attached to the ball faces the point where it is attached to the frame. (There must be enough slack in the wires to allow for what follows.) Next, turn the ball through 180° about a horizontal axis, say the x axis. The vertical wires that were in the direction of the z axis now take on the configuration of Figure 2. Now rotate the ball about its vertical axis. Here the configuration of the z axis wires retains its shape and rotates at half the angular velocity of the ball.

We can go on rotating the ball indefinitely, but it will be found that after every two rotations the system returns to its original configuration. Figure 3 illustrates the sequence of configurations of the model during one such complete cycle.

The wires of this model can be thought of as a set of curvilinear coordinates used to describe the positions of points in a medium that surrounds the spinning core.

For an alternative model, imagine a large spherical canister which is completely filled with a gelatinous medium. Imagine, also, that there is a small magnetized steel ball set in the center. We will presume a reasonable degree of adhesion to exist between the jelly and the walls of the container, and between the jelly and the central ball. By means of external electromagnets we now invert the steel ball by half a turn about the x axis of a coordinate frame centered on the ball. Because of the resilience of the medium this can be effected without tearing or loss of adhesion. Again, using the external electromagnetic field, we set the ball spinning about the

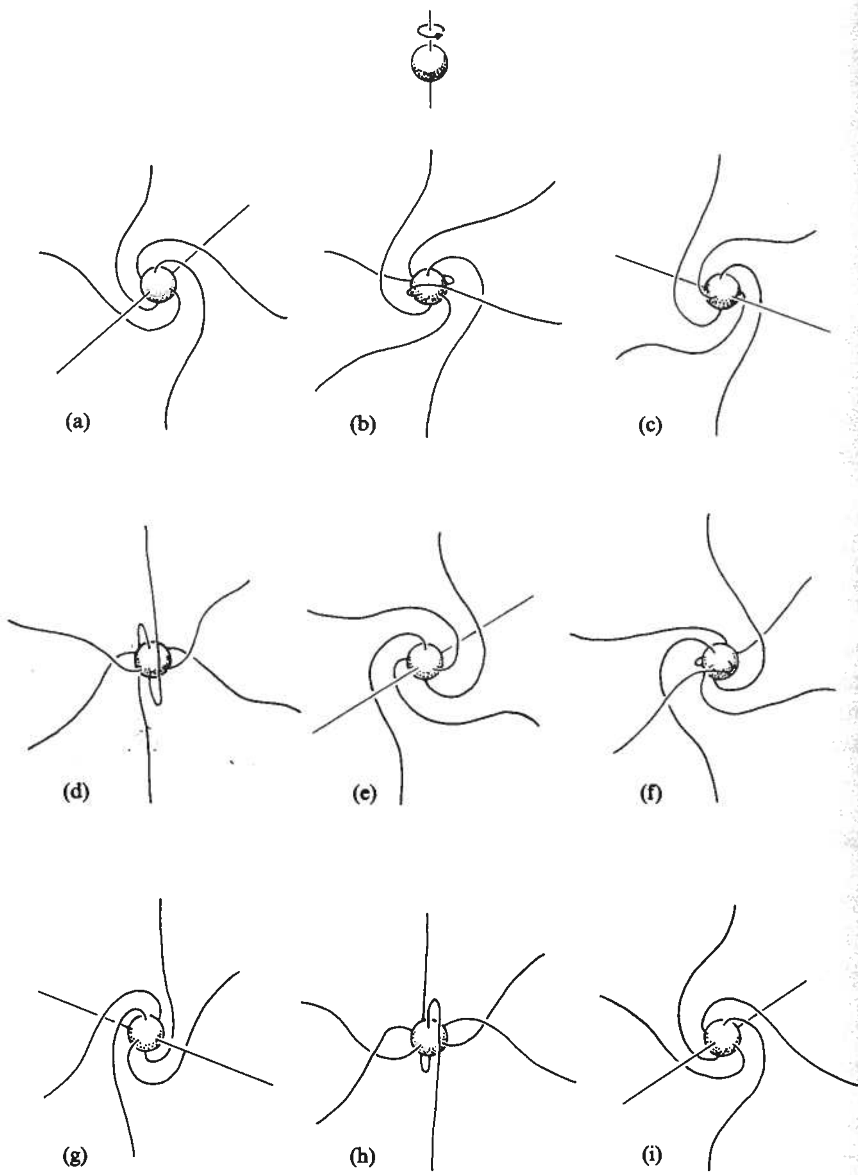

Fig. 3. Top: Rotation of the core is anticlockwise viewed from the upper (positive) z direction. (a) Initial position. (b) After a quarter turn of the core. (c) After half a turn of the core. (d) Three quarters turn of the core. (e) After one full turn of the core. (f) One and a quarter core turns. (g) One and a half core turns. (h) One and three quarter core turns. (i) Two full turns of the core returns the system to the initial position.

z axis. A wave of strain will rotate in the jelly at half the frequency of the steel ball. After every two turns of the steel ball, the system will return to its initial state. The jelly will accommodate the motion without being torn or disrupted in any way.[2]

3. THE MATHEMATICS OF SPHERICAL ROTATION

In the Appendix we have shown that each configuration of the spherical rotation model can be represented by a point on a Euclidean four-dimensional hypersphere (the Lie group space). Taking the hypersphere to be of unit radius, we can describe the transformations from one configuration to another by a closed unimodular group. (Making this choice is, in the language of quantum physics, equivalent to the procedure of *normalizing* the wave function.) If we center this unit hypersphere at the origin of a rectangular Cartesian coordinate system, and let the vector (1, 0, 0, 0) from the origin locate the point corresponding to some chosen initial configuration, then any other configuration will be given by the vector $(\alpha,\beta,\gamma,\delta)$, where $\alpha^2+\beta^2+\gamma^2+\delta^2=1$.[3] A rotation in the spherical mode can be represented by any operator that will transform vectors of this type (or their equivalents) into one another. Here we shall describe two of the most useful representations.

(1) The vector $(\alpha,\beta,\gamma,\delta)$ can be written as the quaternion

$$\phi=\alpha+\mathbf{i}\beta+\mathbf{j}\gamma+\mathbf{k}\delta$$

with $|\phi|^2=\phi^*\phi=\alpha^2+\beta^2+\gamma^2+\delta^2=1$, where ϕ^* is the quaternionic conjugate.

Transformations of this quaternion into any other unimodular quaternion can be effected by multiplying by another suitable quaternion. Thus, unimodular quaternions do duty for both the configuration vector and the rotation operator.

[2]If we replace the canister by a surrounding region of stationary jelly, and the steel ball by a core of spinning jelly, then, insofar as there is strain neither in the stationary region nor in the core, and their potential energies are therefore both zero, we may have an example of a *soliton*. For a general article on Solitons see Claudio Rebbi (1979), p. 92, with further bibliography furnished on p. 168. This might be worth pursuing though we, ourselves, have not done so.

[3]For example, referring to Figure 3, let (1, 0, 0, 0) stand for configuration (a), (0, 1, 0, 0) for configuration (c), then (b) would be $(2^{-1/2}, 2^{-1/2}, 0, 0)$, (d) would be $(-2^{-1/2}, 2^{-1/2}, 0, 0)$, (e) would be $(-1, 0, 0, 0)$ etc.

(2) The quaternion $\alpha+\mathbf{i}\beta+\mathbf{j}\gamma+\mathbf{k}\delta$ can be represented by the column vector

$$\begin{bmatrix}\alpha\\ \delta\\ \gamma\\ \beta\end{bmatrix}$$

This particular arrangement of the coefficients has been chosen to achieve, in what follows, conformity with the traditional formulations of quantum physics.

Now because $\mathbf{i}(\alpha+\mathbf{i}\beta+\mathbf{j}\gamma+\mathbf{k}\delta)=-\beta+\mathbf{i}\alpha-\mathbf{j}\delta+\mathbf{k}\gamma$, left multiplication by $\mathbf{i}$ has the same effect on α, β, γ, and δ as the matrix product:

$$\begin{bmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}\alpha\\ \delta\\ \gamma\\ \beta\end{bmatrix}=\begin{bmatrix}-\beta\\ \gamma\\ -\delta\\ \alpha\end{bmatrix}$$

similarly, $\mathbf{j}$ and $\mathbf{k}$ are, respectively, represented by

$$\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}$$

Thus the full operator $\alpha+\mathbf{i}\beta+\mathbf{j}\gamma+\mathbf{k}\delta$ is represented by

$$\left[\begin{array}{cc|cc}\alpha&-\delta&-\gamma&-\beta\\ \delta&\alpha&\beta&-\gamma\\ \hline \gamma&-\beta&\alpha&\delta\\ \beta&\gamma&-\delta&\alpha\end{array}\right]$$

This matrix can be partitioned as shown. Each of the quadrants is recognizable as the matrix representation of a complex number. Hence we can write

$$\phi=\begin{bmatrix}\alpha+i\delta&-\gamma+i\beta\\ \gamma+i\beta&\alpha-i\delta\end{bmatrix}$$

The determinant of ϕ is $\alpha^2+\delta^2+\gamma^2+\beta^2=1$. This matrix is unitary as well as unimodular or "special". Thus, *this representation of spherical rotation*

TABLE I. Comparison between the Quaternionic and $SU(2)$ Representations of Spherical Rotation

Quaternion: operator and operand	$SU(2)$ operator	Spinor operand	Physical interpretation: Operator	Physical interpretation: Operand
1	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	Leaves the model as it is	Initial position
i	$\begin{bmatrix}0&i\\i&0\end{bmatrix}$	$\begin{bmatrix}i\\0\end{bmatrix}$	Rotates the core of the model through 180° about the x axis (right handedly)	Position attained by rotating core 180° about the x axis from initial position
j	$\begin{bmatrix}0&-1\\1&0\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$	Rotates the core of the model through 180° about the y axis	Position attained by rotating the core 180° about the y axis from initial position
k	$\begin{bmatrix}i&0\\0&-i\end{bmatrix}$	$\begin{bmatrix}0\\i\end{bmatrix}$	Rotates the core of the model through 180° about the z axis	Position attained by rotating core 180° about the z axis from the initial position

consists of the special unitary matrices of order 2, *from which comes the name* $SU(2)$.

The operand form of ϕ is

$$\begin{bmatrix} \alpha + i\delta \\ \gamma + i\beta \end{bmatrix}$$

This form is called a *spinor*, a word coined by P. A. M. Dirac.

Table I shows a comparison between the quaternionic and the $SU(2)$ representations of spherical rotation together with an interpretation of the symbols.

4. SPIN

Rotation that is a linear function of time is referred to as *spin*.

A model of spherical rotation whose initial configuration is given by the spinor $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ can be rotated into the position $\left[\begin{smallmatrix}i\\0\end{smallmatrix}\right]$ by the operator $\left[\begin{smallmatrix}i & 0\\0 & -i\end{smallmatrix}\right]$. This is a rotation of the core of the model by 180° about the z axis. It is represented in the Lie group space by a quarter turn around a great circle in the four-dimensional hypersphere. An intermediate position is given by the spinor

$$\begin{bmatrix} \cos\theta + i\sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} e^{i\theta} \\ 0 \end{bmatrix}$$

where θ is the angular displacement along the great circle. Note that this represents a rotation of the core of the model by 2θ about the z axis. The rotation that brings the model into this position from the initial configuration is represented by the operator $\left[\begin{smallmatrix}e^{i\theta} & 0\\0 & e^{-i\theta}\end{smallmatrix}\right]$. Thus, if we introduce the time parameter, t, then the operator $\left[\begin{smallmatrix}e^{i\omega t} & 0\\0 & e^{-i\omega t}\end{smallmatrix}\right]$ will generate the spin $\left[\begin{smallmatrix}e^{i\omega t}\\0\end{smallmatrix}\right]$ representing core spin of angular velocity 2ω.[4] Similarly,

$$\begin{bmatrix} \cos\omega t & i\sin\omega t \\ i\sin\omega t & \cos\omega t \end{bmatrix}$$

generates a spin in which the core of the model has angular velocity 2ω

[4]The configuration of the z-axis wires of the model (Figure 2) rotates at half the angular velocity of the core, that is, with velocity ω.

about the x axis, and

$$\begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}$$

does the same about the y axis.

In the traditional analysis of spin, it is usual to imply that the process of inverting the axis of a spinning object is identical to reversing the spin. When, however, the spinning object is continuously connected to its stationary environment, this ceases to be true; and we must make a careful distinction between the inversion and the reversal of spin. This distinction affords us insight into one of the most fundamental properties of elementary particles.

To *reverse* the z axis spin $\left[\begin{smallmatrix}e^{i\omega t}\\0\end{smallmatrix}\right]$ one can reverse time $t \to -t$, or one can reverse the angular velocity of the motion: $\omega \to -\omega$. The reversed spin becomes $\left[\begin{smallmatrix}e^{-i\omega t}\\0\end{smallmatrix}\right]$. This motion can be generated by the reverse spin operator $\left[\begin{smallmatrix}e^{-i\omega t} & 0\\0 & e^{i\omega t}\end{smallmatrix}\right]$ from the initial configuration $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. We shall suggestively call this *antispin* and refer to the *antispin state* as opposed to the *normal spin state* of spherical rotation.

To *invert* the spin axis of the core of our model it is necessary to turn the spinning core about one of the axes perpendicular to the spin axis, for example, the y axis. The operator that will achieve this is $\left[\begin{smallmatrix}0 & -1\\1 & 0\end{smallmatrix}\right]$. Therefore, the inverted spin state is given by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^{i\omega t} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{i\omega t} \end{bmatrix}$$

Now,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^{i\omega t} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus, the inverted spin state is generated by the action of the reverse spin operator on the initial configuration $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Lastly, we have the inverted antispin state $\left[\begin{smallmatrix}0\\e^{-i\omega t}\end{smallmatrix}\right]$.

The contrast between the above four states is important for the later understanding of particle physics. However, it must be understood that there exists an infinity of intermediate states of our model. The normal spin state axis can be oriented between the extremes of z axis spin-up and spin-down. The intermediate inverter is

$$\begin{bmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{bmatrix}$$

where 2χ is the axis angle with respect to the spin-up direction. The intermediate spinor is

$$\begin{bmatrix} e^{i\omega t}\cos\chi \\ e^{i\omega t}\sin\chi \end{bmatrix}$$

which is generated by

$$\begin{bmatrix} \cos\omega t + i\sin\omega t\cos 2\chi & i\sin\omega t\sin 2\chi \\ i\sin\omega t\sin 2\chi & \cos\omega t - i\sin\omega t\cos 2\chi \end{bmatrix}$$

from the initial position $\left[\begin{smallmatrix} \cos\chi \\ \sin\chi \end{smallmatrix}\right]$. The corresponding antispin data are obtained by taking the complex conjugate formulas. There is also a continuum of states intermediate between normal spin and antispin. Thus, $\left[\begin{smallmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{smallmatrix}\right]$ operating on the initial configuration $\left[\begin{smallmatrix} \cos\chi \\ \sin\chi \end{smallmatrix}\right]$ will be in the normal spin state for $\chi=0$, and in the antispin state for $\chi=\pi/2$.

5. EQUATIONS OF MOTION OF THE SPHERICAL SPIN MODEL

Suppose we have a model of spherical rotation whose initial configuration is given by the spinor

$$\begin{bmatrix} \alpha + i\delta \\ \gamma + i\beta \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Without loss of generality, we shall suppose spin to be generated by the operator $\left[\begin{smallmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{smallmatrix}\right]$. The center of the spinning core is stationary, and the *phase* of the rotation is, at each instant, the same throughout the space occupied by the model. It will be useful to assume that the stationary framework is large and far away from the core so that we can ascribe the configuration spinor (which describes the phase of the rotation) to every point of the local region.

According to some other observer moving past our system with velocity $-\bar{v}$, the system is represented by the spinor

$$\begin{bmatrix} \phi_1 e^{\frac{i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}} & 0 \\ 0 & \phi_2 e^{\frac{-i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}} \end{bmatrix}$$

Here we have used the Lorentz transformation $t' \to (t-\bar{v}\cdot\bar{r}/c^2)/\beta$, where $\beta=(1-v^2/c^2)^{1/2}$ and $\bar{v}\cdot\bar{r}=v_x x+v_y y+v_z z$. Thus, the moving observer sees the center of our model moving past him with velocity $\bar{v}$ while the phase of the rotation varies not only with time but also from place to place. In fact, from the exponent factor $t-\bar{v}\cdot\bar{r}/c^2$, we can deduce that he sees each *particular* phase of the motion sweeping forward with a velocity of $c^2/|\bar{v}|$ in the direction of $\bar{v}$ (Figure 4). Regions of constant phase form planes perpendicular to the motion of the model.

The observer also reckons that the configuration rotates about *the core* with angular velocity $\omega(1-v^2/c^2)^{1/2}$. This decrease from the value ω is a manifestation of the relativistic principle of time dilatation. On the other hand, this rotation combined with the finite forward motion of the phase produces a configuration helix whose pitch decreases with increasing core

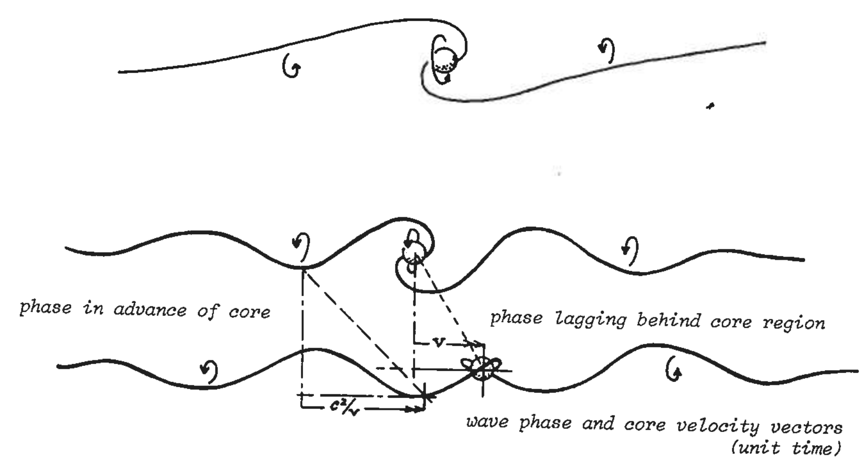

Fig. 4. Top: The motion of the wires of Figure 2. The angular velocity of the core is 2ω and that of the wire configuration is ω. Bottom: The same with its core center moving to the right (in the z direction) with velocity $\bar{v}$. The second figure shows the situation $8\pi/\omega(1-v^2/c^2)^{1/2}$ time units after the first. Note how the wires become helices which rotate contrary to a corkscrew with respect to the advancing core center. (The helix of a corkscrew advances along a fixed locus.)

velocity (exemplified by a select pair of model wires in Figure 4). Measured *at the position of the observer*, this helical configuration rotates with angular velocity $\omega/(1-v^2/c^2)^{1/2}$.[5]

It should be noticed that the mathematical description of the motion of our model describes only the configuration phase: it does not indicate the position of the core center. It follows that a theory entirely based on spinors and their equations will be essentially incomplete.

Within this incomplete theory, we can formulate the general law of motion for the phase of the configuration surrounding the model by deriving the differential equation that is independent of $\bar{v}$:

$$\frac{\partial\phi}{\partial t}=\frac{\partial}{\partial t}\begin{bmatrix}\phi_1 e^{\frac{i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}}\\ \phi_2 e^{\frac{-i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}}\end{bmatrix}=\begin{bmatrix}\phi_1\frac{i\omega}{\beta}e^{\frac{i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}}\\ \phi_2\left(-\frac{i\omega}{\beta}\right)e^{\frac{-i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}}\end{bmatrix}$$

$$=\frac{i\omega}{\beta}\begin{bmatrix}\phi_1 e^{\frac{i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}}\\ -\phi_2 e^{\frac{-i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}}\end{bmatrix}$$

$$\frac{\partial^2\phi}{\partial t^2}=\frac{\partial}{\partial t}\frac{i\omega}{\beta}\begin{bmatrix}\phi_1 e^{\frac{i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}}\\ -\phi_2 e^{\frac{-i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}}\end{bmatrix}=-\frac{\omega^2}{\beta^2}\phi$$

$$\frac{\partial\phi}{\partial x}=-\frac{i\omega v_x}{c^2\beta}\begin{bmatrix}\phi_1 e^{\frac{i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}}\\ -\phi_2 e^{\frac{-i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}}\end{bmatrix}$$

hence

$$\frac{\partial^2\phi}{\partial x^2}=-\frac{\omega^2 v_x}{c^4\beta^2}\phi$$

[5]The different behavior of these two aspects of the angular velocity illustrates the difference between a contravariant and a covariant vector in Minkowski space. $\omega(1-v^2/c^2)^{1/2}$ is, as already mentioned, related to the time component of a contravariant space-time vector; whereas, we shall, when we come to compare our model with an elementary particle, see that $\omega/(1-v^2/c^2)^{1/2}$ is related to the energy of a covariant momentum vector.

Likewise,

$$\frac{\partial^2\phi}{\partial y^2}=-\frac{\omega^2 v_y^2}{c^4\beta^2}\phi \quad\text{and}\quad \frac{\partial^2\phi}{\partial z^2}=-\frac{\omega^2 v_x^2}{c^4\beta^2}\phi$$

Hence, using the notation

$$\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

we have

$$\nabla^2\phi=-\frac{\omega^2}{c^4\beta^2}\left(v_x^2+v_y^2+v_z^2\right)\phi=-\frac{\omega^2 v^2}{c^4\beta^2}\phi$$

Thus,

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=\frac{\omega^2}{c^2}\phi$$

6. FUNDAMENTAL PARTICLES

There have been various attempts to describe fundamental particles as forms of the continuum.[1] But, in conventional thinking, a paradox arises if particles are considered free to spin. For, if there is indeed a continuum that at one point is to be ascribed to the particle, and, at another, to the surrounding void, then the coordinate lines used to map out the whole space at any given instant would, with the passage of time, become twisted up and stretched without limit. Alternatively, they would rip, and one part of the continuum would slide past another along a surface of discontinuity.

Now we maintain that the concept of "ripping the vacuum" is intuitively absurd, and must be rejected, because it introduces discontinuity into a theory whose basic assumption is that we should be able to explain the universe as being continuous. Likewise, an infinite degree of twisting up of the continuum must also be rejected as unworkable. We therefore introduce the following postulate:

Postulate. The only allowable persistent motions of a local region of the continuum with respect to the ambient remainder are such that their motion groups are simply connected and compact.

Under these circumstances, the motion in the continuum will be cyclic, and the system configuration will repeatedly return to each phase of the cycle (see Appendix).

The simplest possible spinning element in the continuum is one that rotates in the spherical mode. Imagine, therefore, that the wires of our model are Gaussian coordinate lines for a reference frame in space. The origin is in a part of space that is spinning (the core) relative to the surrounding parts. The mathematics that we have developed so far is therefore appropriate. The configuration of the model becomes the configuration of space. Here we must sound a cautionary note. The model was described by reference to a rectangular coordinate system in the flat space in which the model was presumed to exist. Now we are assuming that our spinning configuration is the space itself. Use of the model formulas therefore means that we shall be describing real space by reference to a nonexistent background. This will work for the same reason that enabled Newton to develop a theory of gravitation on the assumption that space was everywhere Euclidean. But, as was the case with Newton's theory, our theory may require later refinement.[6]

Our conclusion, then, is that the spinning continuum is surrounded by an undulating, wavelike region whose phase, ϕ, satisfies the equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\omega^2}{c^2}\phi$$

This looks very like the Klein–Gordon equation for an elementary particle. We shall therefore make the identification exact. The usual formulation of the Klein–Gordon equation is

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{m^2c^2}{\hbar^2}\Psi$$

where m is the mass of the particle, $\hbar$ is Plank's constant divided by 2π, and Ψ is the wave function for the particle. The wave function must, therefore, be identified with the spinor that designates the *phase* of the spherical spin in space and time. We can now see why the wave function Ψ has no physically meaningful magnitude, but is usually normalized. The identification requires that we put

$$\frac{m^2c^2}{\hbar^2} = \frac{\omega^2}{c^2}$$

whence

$$|m| = \frac{\hbar}{c^2}|\omega|$$

[6]For example, a radially dependent gauge transformation can be added to an exact theory of particles (that includes a description of the core location) to bring about a contraction in the volume of the particle's space. This could account for the gravitational field.

Thus, we deduce that when we have a spinning region of the continuum, it interacts with other spinning regions in such a way that we find it necessary to introduce a measure for the inertia of the interactions. We arbitrarily define the unit of mass. But it would appear that the inertia of the spinning region is simply a manifestation of its angular velocity; and, if this is measured in radians per second, then the ratio of the arbitrary mass unit to the spin frequency is $\hbar/c^2$, which must, therefore, be a constant. This is the significance of Plank's constant.

The ascription of mass to the angular velocity of the spin is relative to the observer and measured at his location.[7] So a particle, whose mass is m_0 when stationary, has a mass $m_0/(1-v^2/c^2)^{1/2}$ when in motion with velocity $\bar{v}$ (see page 458). Another consequence of the identification $mc^2=\hbar\omega$ is that the configuration spin, that is, the outermost undulations of the system, is the *de Broglie wave*. This wave embodies a core region that, in the stationary particle, spins with frequency ω/π. This is the Zitterbewegung that was first studied by E. Schrödinger (1930). In the moving particle, the Zitterbewegung frequency slows to $(\omega/\pi)(1-v^2/c^2)^{1/2}$ (see page 458).[8]

To increase our insight into the elementary particle formed by the spherically spinning manifold, we shall establish a first-order differential equation for the spinor as is done in relativistic quantum mechanics. The standard first-order operator in Minkowski space is the *quad*, $\Box$, whose second-order derivative is the d'Alembertian $\Box^2=\nabla^2-\partial^2/c^2\partial t^2$. The Klein–Gordon equation is often written in the form

$$(\Box^2 - m^2c^2/\hbar^2)\Psi = 0$$

The quad operator, as normally used, is a four-dimensional vector operator. We shall modify it so that it becomes a spinor operator. In traditional quantum mechanics it is usual, at this stage, to introduce a group of Hermitian matrices known as Pauli's spin matrices. We shall not do that

[7]We shall use the word "observer" in the way that is usual when discussing different reference frames. At the level of an elementary particle, the notion of a real observer ceases to have meaning. We shall retain the word, however, as meaning the person or object (in this context, the other interacting particle) to whom the relevant calculations apply.

[8]In traditional quantum theory, the mathematical formalism that produces the Zitterbewegung term indicates a frequency that increases with the particle's motion. This is because the formalism is designed to predict the results of actual measurements. A measurement made on a moving system is made with reference to the stationary framework of the measuring device. It is the *projection* of the spin of the moving core onto the stationary measuring frame that increases in frequency. Most texts on relativistic quantum theory go through the exercise of using the established mathematical formalism to calculate a particle's velocity. We give, as an example, J. McConnell (1960).

here as we have no reason to use Pauli's matrices and no interpretation for them. We shall therefore introduce the operator

$$\square=\begin{bmatrix}1&0\\0&1\end{bmatrix}\frac{\partial}{ic\partial t}+\begin{bmatrix}0&i\\i&0\end{bmatrix}\frac{\partial}{\partial x}+\begin{bmatrix}0&-1\\1&0\end{bmatrix}\frac{\partial}{\partial y}+\begin{bmatrix}i&0\\0&-i\end{bmatrix}\frac{\partial}{\partial z}$$

We have a physical interpretation for each term in this expression. The matrices used form a basis for the algebra of $SU(2)$. Each of the space term matrices turns the derived spinor in the term through a quarter turn (or the core through a half turn) about the associated axis. The quaternionic form of the operator is

$$\frac{\partial}{ic\partial t}+\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$$

which is strongly suggestive of the usual quad operator. The quaternionic form reminds us that there is a conjugate operator

$$\square^*=\begin{bmatrix}1&0\\0&1\end{bmatrix}\frac{\partial}{ic\partial t}-\begin{bmatrix}0&i\\i&0\end{bmatrix}\frac{\partial}{\partial x}-\begin{bmatrix}0&-1\\1&0\end{bmatrix}\frac{\partial}{\partial y}-\begin{bmatrix}i&0\\0&-i\end{bmatrix}\frac{\partial}{\partial z}$$

We note that $\square^*\square=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]\square^2$, where $\square^2$ is the d'Alembertian operator.

Let us now calculate $\square\phi$, where ϕ is the spinor for spherical spin in the manifold. To save space, since the exponential parts of the spinors occur so frequently, we shall introduce the abbreviations

$$e^+=e^{\frac{i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}},\qquad e^-=e^{\frac{-i\omega(t-\bar{v}\cdot\bar{r}/c2)}{\beta}}$$

Using the derivatives from page 457, we see that

$$\square\phi=\frac{\omega}{c\beta}\begin{bmatrix}\phi_1e^+\\-\phi_2e^-\end{bmatrix}+\frac{\omega v_x}{c^2\beta}\begin{bmatrix}-\phi_2e^-\\\phi_1e^+\end{bmatrix}+\frac{\omega v_y}{c^2\beta}\begin{bmatrix}-i\phi_2e^-\\-i\phi_1e^+\end{bmatrix}+\frac{\omega v_z}{c^2\beta}\begin{bmatrix}\phi_1e^+\\\phi_2e^-\end{bmatrix}$$

$$=\frac{\omega}{c}\begin{bmatrix}\dfrac{1+v_z/c}{\beta}\phi_1e^+-\dfrac{(v_x+iv_y)/c}{\beta}\phi_2e^-\\[2ex]\dfrac{(v_x-iv_y)/c}{\beta}\phi_1e^+-\dfrac{1-v_z/c}{\beta}\phi_2e^-\end{bmatrix}$$

or

$$\square\phi=\frac{\omega}{c}\phi' \tag{A}$$

where we have defined the quantity

$$\phi'=\frac{1}{\beta}\begin{bmatrix}(1+v_z/c)&-(v_x+iv_y)/c\\(v_x-iv_y)/c&-(1-v_z/c)\end{bmatrix}\phi$$

Now we have already seen that $\square^*\square\phi=\square^2\phi=(\omega^2/c^2)\phi$. Hence

$$\square^*\frac{\omega}{c}\phi'=\frac{\omega^2}{c^2}\phi$$

or

$$\square^*\phi'=\frac{\omega}{c}\phi \tag{B}$$

We can combine (A) and (B) into one equation by writing

$$\tilde{\phi}=\begin{bmatrix}\phi_1e^+\\\phi_2e^-\\\dfrac{(1+v_z/c)}{\beta}\phi_1e^+-\dfrac{(v_x+iv_y)}{c\beta}\phi_2e^-\\[2ex]\dfrac{(v_x-iv_y)}{c\beta}\phi_1e^+-\dfrac{(1-v_z/c)}{\beta}\phi_2e^-\end{bmatrix}$$

Then

$$\left[\begin{array}{cc|cc}1&0&0&0\\0&1&0&0\\\hline0&0&1&0\\0&0&0&1\end{array}\right]\frac{\partial\tilde{\phi}}{ic\partial t}+\left[\begin{array}{cc|cc}0&i&0&0\\i&0&0&0\\\hline0&0&0&-i\\0&0&-i&0\end{array}\right]\frac{\partial\tilde{\phi}}{\partial x}$$

$$+\left[\begin{array}{cc|cc}0&-1&0&0\\1&0&0&0\\\hline0&0&0&1\\0&0&-1&0\end{array}\right]\frac{\partial\tilde{\phi}}{\partial y}+\left[\begin{array}{cc|cc}i&0&0&0\\0&-i&0&0\\\hline0&0&-i&0\\0&0&0&i\end{array}\right]\frac{\partial\tilde{\phi}}{\partial z}$$

$$=\frac{\omega}{c}\left[\begin{array}{cc|cc}0&0&1&0\\0&0&0&1\\\hline1&0&0&0\\0&1&0&0\end{array}\right]\tilde{\phi}$$

The top left quarter partition of the left-hand side of the equation, being the statement $\Box\phi$, is equated to the lower half partition of $(\omega/c)\tilde{\phi}$ by means of the partition-reversing operator

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The question naturally arises, *is this equivalent to Dirac's equation*? To see that it is, we write $\tilde{\Psi}$ in place of $\tilde{\phi}$, $mc/\hbar$ in place of ω/c, multiply by $\hbar c$, extract the factor i from the space terms, and rearrange to form

$$\frac{\hbar c}{i}\left\{\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\frac{\partial\tilde{\Psi}}{\partial x}+\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}\frac{\partial\tilde{\Psi}}{\partial y}\right.$$

$$\left.+\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\frac{\partial\tilde{\Psi}}{\partial z}\right\}+\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}mc^2\tilde{\Psi}$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}i\hbar\frac{\partial\Psi}{\partial t}$$

We can now introduce a similarity transformation. The matrix

$$A=A^{-1}=\begin{bmatrix} -2^{-1/2} & 0 & 2^{-1/2} & 0 \\ 0 & -2^{-1/2} & 0 & 2^{-1/2} \\ 2^{-1/2} & 0 & 2^{-1/2} & 0 \\ 0 & 2^{-1/2} & 0 & 2^{-1/2} \end{bmatrix}$$

is its own inverse. This gives the following transformations:

$$AIA^{-1}=I,\qquad A\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}A^{-1}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(It was this result that dictated the form of A.)

$$A\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}A^{-1}=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$A\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}A^{-1}=\begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix},$$

$$A\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}A^{-1}=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Hence our spinning manifold equation becomes

$$\frac{\hbar c}{i}\left\{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\frac{\partial\Psi}{\partial x}+\begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}\frac{\partial\Psi}{\partial y}\right.$$

$$\left.+\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\frac{\partial\Psi}{\partial z}\right\}+\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}mc^2\Psi$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}i\hbar\frac{\partial\Psi}{\partial t}$$

where $\Psi=A\tilde{\Psi}$.

This is Dirac's equation. Thus, we have shown that the entity formed by a spherically rotating disturbance of the manifold is a Dirac particle.

Evaluating Ψ, we have

$$\Psi=\begin{bmatrix} -2^{-1/2} & 0 & 2^{-1/2} & 0 \\ 0 & 2^{-1/2} & 0 & 2^{-1/2} \\ 2^{-1/2} & 0 & 2^{-1/2} & 0 \\ 0 & 2^{-1/2} & 0 & 2^{-1/2} \end{bmatrix}$$

$$\times\begin{bmatrix} \phi_1 e^+ \\ \phi_2 e^- \\ \dfrac{(1+v_z/c)}{\beta}\phi_1 e^+ - \dfrac{(v_x+iv_y)}{c\beta}\phi_2 e^- \\ \dfrac{(v_x-iv_y)}{c\beta}\phi_1 e^+ - \dfrac{(1-v/c)}{\beta}\phi_2 e^- \end{bmatrix}$$

$$=\begin{bmatrix} \left(-\dfrac{1}{2^{1/2}}+\dfrac{1+v_z/c}{2^{1/2}\beta}\right)\phi_1 e^+ - \left(\dfrac{v_x+iv_y}{2^{1/2}\beta c}\right)\phi_2 e^- \\ \left(\dfrac{v_x-iv_y}{2^{1/2}\beta c}\right)\phi_1 e^+ - \left(\dfrac{1}{2^{1/2}}+\dfrac{1-v_z/c}{2^{1/2}\beta}\right)\phi_2 e^- \\ \left(\dfrac{1}{2^{1/2}}+\dfrac{1+v_z/c}{2^{1/2}\beta}\right)\phi_1 e^+ - \left(\dfrac{v_x+iv_y}{2^{1/2}\beta c}\right)\phi_2 e^- \\ \left(\dfrac{v_x-iv_y}{2^{1/2}\beta c}\right)\phi_1 e^+ + \left(\dfrac{1}{2^{1/2}}-\dfrac{1-v_z/c}{2^{1/2}\beta}\right)\phi_2 e^- \end{bmatrix}$$

For the stationary particle, $\bar{v}=0$, and

$$\Psi=\begin{bmatrix} 0 \\ -2^{1/2}\phi_2 e^{-i\omega t} \\ 2^{1/2}\phi_1 e^{i\omega t} \\ 0 \end{bmatrix}$$

The factor $2^{1/2}$ is unimportant. We started with the normalized 2-spinor $\begin{bmatrix}\phi_1 e^+ \\ \phi_2 e^-\end{bmatrix}$, and have derived what is called a 4-spinor, Ψ, which is not normalized. The normalization of the 2-spinor was arbitrary, and was based upon selecting a unit hypersphere for $SU(2)$. Had we chosen a hypersphere of radius $2^{-1/2}$, then our 4-spinor would have magnitude $\Psi^*\Psi=1$, where Ψ^* is the conjugate transposed row matrix. We could then absorb the $2^{1/2}$ factors by writing $\psi_1=2^{1/2}\phi_1$ and $\psi_2=2^{1/2}\phi_2$ to produce a normalized 4-spinor,

$$\begin{bmatrix} 0 \\ -\psi_2 e^{-i\omega t} \\ \psi_1 e^{i\omega t} \\ 0 \end{bmatrix}$$

in which $\psi_1{}^*\psi_1+\psi_2{}^*\psi_2=1$.

Now if we had begun by assuming our manifold to be in the state described by $\begin{bmatrix}\phi_1 e^- \\ \phi_2 e^+\end{bmatrix}$, and repeated the previous calculations, we would have found the corresponding 4-spinor for the system at rest to be

$$\begin{bmatrix} -\psi_1 e^{-i\omega t} \\ 0 \\ 0 \\ \psi_2 e^{i\omega t} \end{bmatrix}$$

All other spin directions also satisfy the Dirac equation. For example, y axis spin represented by the spinor

$$\begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

(see page 455) corresponds to the renormalized Dirac spinor

$$\frac{1}{2}\begin{bmatrix} (-\psi_1+i\psi_2)e^{-i\omega t} \\ (-i\psi_1-\psi_2)e^{-i\omega t} \\ (\psi_1+i\psi_2)e^{i\omega t} \\ (-i\psi_1+\psi_2)e^{i\omega t} \end{bmatrix}$$

for a particle at rest. The most general Dirac 4-spinor for a particle at rest is

$$\frac{1}{2}\begin{bmatrix} \{(-1+n)\psi_1+(im+l)\psi_2\}e^{-i\omega t} \\ \{(-im+l)\psi_1+(-1-n)\psi_2\}e^{-i\omega t} \\ \{(1+n)\psi_1+(im+l)\psi_2\}e^{i\omega t} \\ \{(-im+l)\psi_1+(1-n)\psi_2\}e^{i\omega t} \end{bmatrix}$$

being derived from the 2-spinor

$$\begin{bmatrix} \cos\omega t + in\sin\omega t & (-m+il)\sin\omega t \\ (m+il)\sin\omega t & \cos\omega t - in\sin\omega t \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

where l, m, n, are the direction cosines of the spin axis.

7. PARTICLE STATES

Earlier (page 455) we selected a particular combination of initial configuration and spin axis of our rotational mode and labeled it the "normal spin state" in order to contrast it with the "antispin state" obtained by reversing the spin. We also defined "inversion" of those states from the "up" to the "down" orientation. We shall now compare those labeled spin states with their corresponding 4-spinors.

By comparing these results (Table II) with those of standard quantum theory, we see that what we have called the "antispin states" produce solutions that are normally applied to the electron. Our "normal spin states" therefore correspond to the positron. *The theory is symmetrical.* We no longer need to postulate the existence of holes in a sea of negative energy states for the positron as is done in standard theory. We assert that all mass and energy are cyclical disturbances of the continuum, and that their measures are proportional to their frequencies. There is, therefore, no such thing as negative energy any more than there is negative temperature, in the absolute sense.

There is no immediately obvious reason why the spinning continuum must, in practice, be either a particle or an antiparticle. Presumably, some basic restriction forbids the continuum of intermediate states in free particles suggested by the formula at the end of section 4 (page 456). It would appear that the rotation axis is locked into the configuration at any given instant. Of significance is the fact that the antispin states are mirror images of the corresponding normal spin states (this can be seen by examining the model of spherical rotation).

We have produced a theory of particles and their states that, contrary to traditional quantum theory, is independent of measurements made on the particles. A particle, as such, is unobservable (see Appendix, page 456). Measurements can only be made by interacting with the particle, and the only tools for investigation are other regions of distorted continuum (we have here made the presumptive leap of assuming photons to be undulations of the space-time structure). Interactions will be governed by rules of wave interference and by whatever geometric conservation laws turn out to be fundamental laws of the universe.

TABLE II. Comparison of 2-Spinor and 4-Spinor Spin States

Name of state	2-Spinor	4-Spinor
Normal spin "up"	$\begin{bmatrix} e^{i\omega t} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ e^{i\omega t} \\ 0 \end{bmatrix}$
Normal spin "down"	$\begin{bmatrix} 0 \\ e^{i\omega t} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{i\omega t} \end{bmatrix}$
Antispin "up"	$\begin{bmatrix} e^{-i\omega t} \\ 0 \end{bmatrix}$	$\begin{bmatrix} -e^{-i\omega t} \\ 0 \\ 0 \\ 0 \end{bmatrix}$
Antispin "down"	$\begin{bmatrix} 0 \\ e^{-i\omega t} \end{bmatrix}$	$\begin{bmatrix} 0 \\ -e^{-i\omega t} \\ 0 \\ 0 \end{bmatrix}$

The negative coefficients of the antispin 4-spinors are of no significance and can be eliminated by a time shift of π/ω.

It is clear that, in order to find the position of the particle, it is necessary to locate the center of the spinning core. This means that the whole disturbance that constitutes the particle has to be confined. This confining process, which is implemented by other manifold disturbances, consists in so enriching the harmonic content of the particle's de Broglie wave that its momentum becomes unpredictably altered. *Thus, the measuring process is limited by the indeterminancy principle*—notwithstanding which, the particle does at all times have an exact location, and, when completely free of interactions with other particles, has a very definite momentum.

Implied by the principles that have been formulated in this paper is the idea that there can be more complex disturbances that do not disrupt the continuity of space. An atom in a given energy state is probably an example of this higher-order complexity. This being so, the same principle that prevents spherical spin from exchanging rotational energy with its surroundings would also inhibit the atom from radiating energy. This allows for the existence of steady states of atoms within the structure of a classical theory of the continuum, and thus renders Neils Bohr's *ad hoc* hypothesis unnecessary.

APPENDIX

A.1 LIE GROUPS AND THREE-DIMENSIONAL ROTATION[9]

In order to analyze the nature of three-dimensional rotation, we shall construct what is called the *Lie group* space for the rotation.

Let S be a ball free to rotate about its center C. Let the initial position of S be represented in the Lie group space by the point O. If S is rotated through angle θ about an axis l, its new position will be designated by the point in the Lie group space that lies θ units from O on the line through O parallel to the axis l, and in the direction that a right-handed corkscrew would move if it were placed along l and turned through the angle θ. A rotation θ in the reverse direction about l is represented by a point θ units from O but in the opposite direction from the previous point. Because a rotation of π units about l will take the ball S into exactly the same position as the counterrotation $-\pi$, it follows that in the Lie group space, we need only go as far as π units from O in each of the directions parallel to l in order to end up with a linear array of points representing every possible position of S that leaves l fixed. We can, therefore, identify the two end points π and $-\pi$ as one and the same point. This construction is repeated for all the possible orientations of the axis l.

The Lie group space, then, consists of a "sphere" of radius π units centered on O with diametrically opposite points identified. The name of this group is the *proper orthogonal group in three dimensions* and is denoted by the symbol $O(3)+$. It is also sometimes called the three-dimensional rotation group, but this is a misnomer as we shall see in due course.

A Lie group is, from a mathematical standpoint, a fairly complicated structure. It can be thought of as being a compatible combination of more basic structures. Thus it is, amongst other things, a *topological* space, and for our purposes it will be sufficient to describe a topological space as the vehicle for those elements of geometry that are unaffected by plastic deformations—such as the continuity or *connectedness* of the space: a property that is related to its *homotopy type*.

In a topological space, the paths between two points are said to be *homotopic* if the one path can be continuously deformed into the other. It is understood that all points on the path remain points of the topological space during the transition. If the initial and terminal points of a path happen to coincide, it becomes a *closed path*. By examining the closed paths that begin and end on a particular point, we can gain some idea about the total or *global* structure of the space.

[9]D. Speiser (1964). This article was the precursor of our theory.

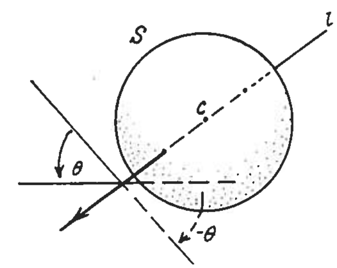

Fig. 5. The rotating object under consideration.

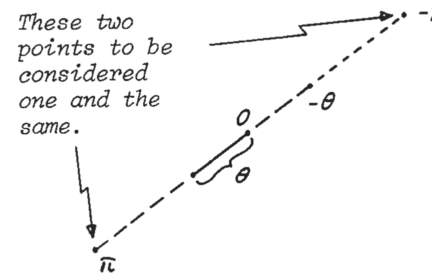

Fig. 6. The Lie group space of the rotations.

As an illustration, let us take the two-dimensional space of the surface of a torus (a figure shaped like an anchor ring or doughnut). Figure 9 is a picture of it. Emanating from the point P we see two paths, p_1 and p_2, that can clearly be deformed into one another; that is to say, they are homotopic. The collection of all closed paths through P that are homotopic to one another forms a *homotopic class*. p_1 and p_2 are thus in the same homotopic class. The paths p_3 and p_4, on the other hand, cannot be deformed into one another or into p_1 or p_2; they are members of two other homotopic classes. The first class mentioned, that of p_1 and p_2, has the unique feature of including the degenerate path, namely, the point P itself. Paths like p_1 and p_2 can be contracted continuously until only the point P remains. The point P, thought of as a degenerate path, is called a *null path*. The set of all homotopic classes together with their interrelationships (they form a mathematical *group*) characterize the space and designate its homotopy type.

Of special importance to Lie group spaces is the type in which there is only one homotopic class associated with each point. This class necessarily

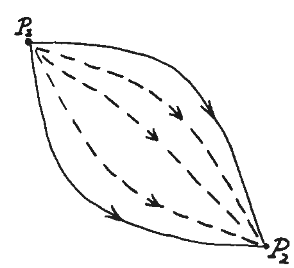

Fig. 7. Two paths which can be continuously deformed into one another are homotopic.

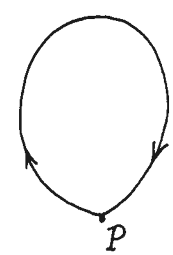

Fig. 8. A closed path.

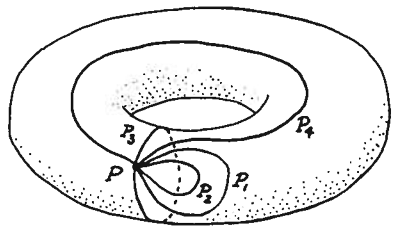

Fig. 9. Paths on the surface of a torus.

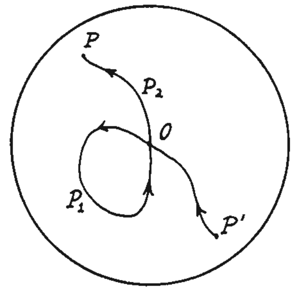

Fig. 10. The Lie group space of $O(3)+$.

includes the null path: in other words, every closed path can be contracted into a point. A topological space of this type is described as being *simply connected.* If the topological space is also a Lie group, then it is called a *universal covering group*, because there is only one simply connected Lie group with a given local structure, and because all other Lie groups having the same local structure, but which are not simply connected, are *homomorphisms* of it.

These properties and definitions will now be exemplified by returning to the discussion of the orthogonal group in three dimensions. We recall that the group space of $O(3)+$ was a sphere of radius π with diametrically opposite points identified.

Consider the closed paths that begin and end at O in Figure 10. A path p_1 that remains everywhere within the sphere can be contracted into the null path at O. But what of a path that crosses the boundary of the sphere? Let p_2 be the locus of a point that starts from O and passes out through the surface of the sphere at point P. Since P is the same as the diametrically opposite point P', the continuation of this path is traced by a point moving inwards from the point P'. If this path finally returns to O, we have a closed path. Can this path be continuously contracted into the null path? The answer is no. Any attempt to so contract p_2 must, at some stage, reduce the path to one that is everywhere within the boundary of the sphere, like p_1. But in order to do that smoothly and continuously, the point P must sooner or later move closer to P' until they meet. However, this is impossible, because P' is always diametrically opposite P. Thus, all the paths that cut the boundary *once* and then return to O form a homotopic class distinct from the class that includes the null path. Hence, we have shown that $O(3)+$ is not simply connected.

There does, however, exist a Lie group space that is locally the same as $O(3)+$, but which *is* simply connected. For, consider two spheres of

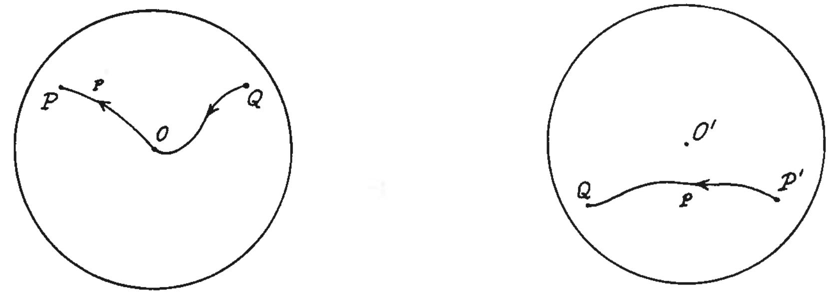

Fig. 11. The Lie group space of $SU(2)$, which is simply connected, but locally the same as $O(3)+$.

radius π centred on O and O', both replicas of $O(3)+$ except that a point on the boundary of the first sphere becomes identified with the point on the second sphere that corresponds to its diametrically opposite position (Figure 11). In this space, we can trace a path p from O that leaves the first sphere at a boundary point P and reappears entering the second sphere from P'. After crossing the second sphere, the path leaves at a point Q', and reappears in the first sphere at Q, whence it returns to O. It is now possible to contract the path p towards O, since we are free to allow P' to approach Q' (and, simultaneously, P must move towards Q). Any closed path through O can, therefore, be contracted into the null path. Hence, the space is simply connected. This Lie group space is the universal covering group of $O(3)+$, and is called $SU(2)$ for reasons that were given on page 454.

A.2 PHYSICAL INTERPRETATION (Bolker, 1973)

We recall that the group space $O(3)+$ was a geometrical description of the rotations of a ball, S, in which every position of S was represented uniquely by a point in that space. The initial position of S was represented by the point O.

Let us find the physical meaning of traversing a small closed path through O. As we leave O in a given direction, the ball S turns about the axis parallel to that direction. If we now travel across the Lie group space keeping roughly the same distance from O, the ball S will rotate about various axes in such a way that it remains at roughly the same angular deviation from the initial position. Finally, as we return to O, the ball decreases its angular deviation until it is right back in the position from which it started.

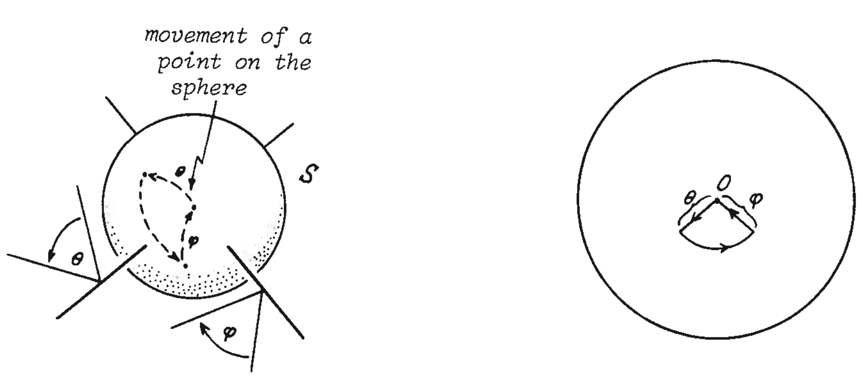

Fig. 12. A small wobble of the sphere S picks out a small path through O in the group space $O(3)+$.

How would one describe the motion executed by S? Clearly, the answer is that S wobbled; that is to say, it executed the sort of motion that could have been imparted to the ball if it were held firmly in the hand, the wrist and elbow given a few flexes and twists, and then relaxed back into the original position.

We can now give a physical interpretation of homotopy. A closed path on O that is just a shade displaced from the previous path represents a wobble of S in which the motions occur around axes that are slightly displaced from the previous axes, and to an extent that is a shade different. The transition from one path to the other, then, consists of either exaggerating or diminishing the motion. Thus, the sequence of paths in the Lie group space constituting the smooth transition between two homotopic paths requires that the motions of the ball S underlying the one path be capable of smooth and gradual exaggeration until they coincide with the motions underlying the final path. All the paths of the homotopic class that includes the null path are describable as wobbles of a greater or lesser extent; that is, they can be demonstrated with S held firmly in the hand.

In $O(3)+$, a closed path that leaves O, that moves straight along a radius until it leaves the surface of the space, that reappears at the diametrically opposite point, and then moves straight back to O corresponds to one complete revolution of S around an axis parallel to the path. This path is not in the same homotopic class as the null path. In $SU(2)$, however, the Lie group space of $O(3)+$ is represented twice. A closed straight path that leaves O along a radius crosses the second space via the diameter through O' before reappearing in the first space and returning to O. The ball S, in this case, makes *two* complete turns around an axis parallel to the path. But $SU(2)$ is simply connected; therefore, the path

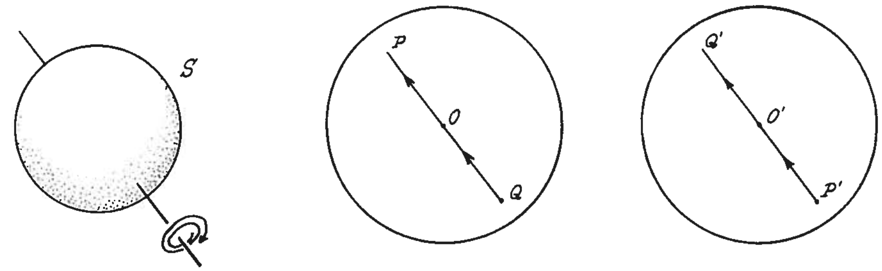

Fig. 13. A closed path in $SU(2)$ consisting of two parallel diameters through O and O' represents a double rotation of S.

described is homotopic to, and, so, contractible into the null path. This means that the motion of S (i.e., a complete double rotation around an axis) can be arrived at by a smooth exaggeration of a wobble, which, in turn, means that it is possible to grasp a ball in the hand, and, by flexing of wrist and elbow joints, etc., to execute a complete double (but not single) revolution of the ball about a fixed axis, and thereby end up in the same stance as one began.

This is, in fact, the basis of an old party trick in which the joker twirls a bowl of soup around its vertical axis so as not to spill a drop. The trick requires raising and lowering the bowl somewhat during the gyration to accommodate the awkward jointing of the arm; but it is, nevertheless, an illustration of the above theory. The bowl makes two full turns in executing the complete gyration.

We can replace the human arm and the soup bowl by a long coiled spring and the ball, S. The top of the spring can be clamped to a stationary framework in imitation of the shoulder connection, and the bottom end

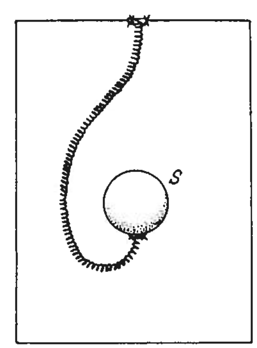

Fig. 14. Coiled spring simulating a human arm.

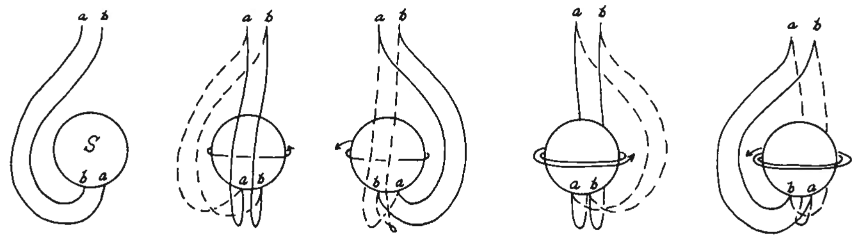

Fig. 15. Each half turn of S forms a twist in the two tracers (dotted lines), which can only be undone by the displacement shown (solid lines).

can be bent around and brought up underneath the ball in imitation of the hand (Figure 14). The uniform flexibility of the spring allows the ball to spin without the awkward vertical movement that occurs in the soup bowl trick.

The motion of the ball and spring can be understood in detail by considering two lines drawn opposite one another down the length of the unstrained spring. The spring is forced to bend in assuming the above configuration, but it will swing into the plane that minimizes its torsion. Examination of the diagrams of Figure 15 shows that the configuration of the spring will rotate at half the angular velocity of the ball.[10]

Now, because of the shape of this system, we can add another spring symmetrically opposite to the one above. It, too, must advance at half the angular velocity of the ball and will thereby maintain its relationship with the first spring (Figure 16). The question naturally arises: how complicated a system of springs can participate in the motion? To answer this, we go back to first principles.

Consider a ball to which is attached a large indefinite number of long narrow springs running radially outwards and tied at their outer ends to a fixed framework (Figure 17). Since the springs are flexible, it is possible for the ball to execute small wobbles about its center without causing any

[10]Mentioned in Bolker (1973) are both Dirac's spanner—a demonstration of spherical rotation using strings attached to an asymmetric wrench—and, also, D. A. Adams's patented device for transmitting electricity to a rotating turntable without the use of slip rings. The Adams device is identical in its symmetry to our incipient model. A full and well-illustrated report on it appeared in Adams (1975). Just for the record, we would like to point out that our model depicted in Figure 1 was constructed and demonstrated in the Mathematics Department at Queen's University, Ontario, Canada in the fall of 1966. E. P. Battey-Pratt would like to acknowledge his indebtedness to Prof. A. J. Coleman for inspiration and to Hans Kummer who, in conversation, helped clarify the role of universal covering groups in the theory.

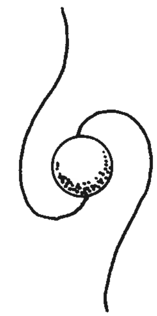

Fig. 16. A symmetric pair of springs.

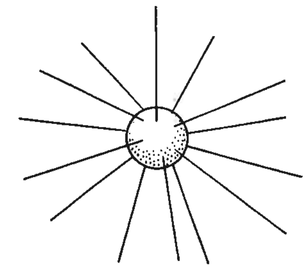

Fig. 17. Rays of springs emanating from a ball.

entanglement of the springs. But we have seen that a double rotation of the ball can be developed by the continuous exaggeration of a small wobble. Now, the transition from a motion that does not cause the springs to tangle and knot up to one that does is necessarily abrupt and discontinuous, and, therefore, cannot occur during our continuous development of the wobble into a double rotation.

Thus, we see that the springs in Figure 16 are but two of an infinite number of "rays" that can participate in the motion of the ball without knotting up. To find the disposition of the other rays, we simply note that, starting from the model of Figure 17 we must give the ball a half turn about a horizontal axis in order to put the vertical springs into the configuration of Figure 16. The ball may then be rotated about its vertical axis. This is the explanation for the validity of the models of Section 2 of the paper (page 448).

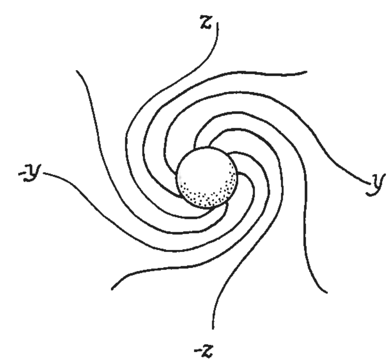

Fig. 18. Springs in the plane perpendicular to the initial half turn about the x axis.

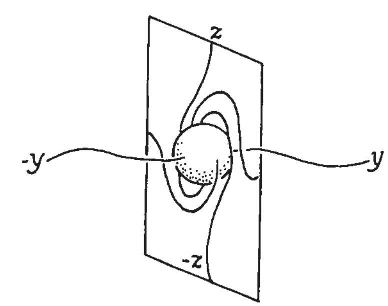

Fig. 19. As the ball is rotated about the z axis, the axis of the initial half turn also rotates.

Now let us go back and follow through the same arguments for the case of two-dimensional rotation. We shall then produce a geometrical analogy that will illuminate the relationship between the above analysis and the conventional view of three-dimensional rotation.

A.3 TWO-DIMENSIONAL ROTATION

The Lie group space for the two-dimensional rotation of a flat object about some point in the plane is very simple. For convenience, let the flat object be a circular disk D rotating about its center. Let the initial position of the disk be represented by the point O in the Lie group space. A counterclockwise rotation of D through an angle θ is represented by a point θ units to the right of O in the group space. Similarly, a clockwise rotation, $-\theta$, of D is represented by a point θ units to the left of O. Since a rotation π brings D to the same position as the counterrotation $-\pi$, we can identify these two points. Thus, the Lie group space is a line of length 2π units with end points identified. This may be thought of as being the circumference of a circle with O and the dual point π, $-\pi$ diametrically opposite. The name of this Lie group is *the proper orthogonal group in two*

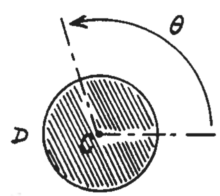

Fig. 20. The rotating disk.

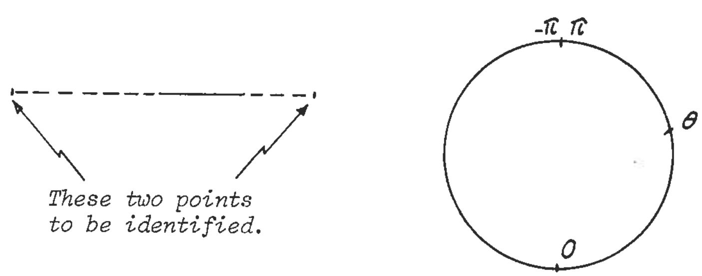

Fig. 21. The Lie group space $O(2)+$.

Fig. 22. The group space R derived from $O(2)+$.

dimensions, or $O(2)+$. It is often referred to, erroneously, as the two-dimensional rotation group.

Since the Lie group space is one-dimensional, the only type of path that is homotopic to the null path through O is one that leaves O, goes to some other point, and then returns to O again by tracking back along its outward path. $O(2)+$ is, therefore, not simply connected since it is possible, by completing one or more circuits of the space, to return to O without back tracking.

In order to construct the universal covering group of $O(2)+$ (the locally isomorphic but simply connected group space), we duplicate the space as we did for $O(3)+$, and make the end point of the original space conterminal with the beginning point of the repeated segment rather than terminating with the other end of itself. We see, however, that we still cannot rejoin the other end point of the repeated space to the open end of the original because to do so would produce a circle, which is not simply connected. We have, therefore, to repeat the segment once more. Again, we come to an end that cannot be joined back to the first segment without creating a circular, noncontractible path. Thus, the repetitions of the segment never end, and the Lie group space of $O(2)+$ must be extended indefinitely in repetitions of itself in either direction. The universal covering group of $O(2)+$ is, therefore, an infinite line. Because every point on an infinite line can be represented by a real number, the symbol for this group is R. It is the same as the Euclidean one-dimensional continuum.

A closed path in R is effectively just a line segment that is traversed both ways, both out and back again to the starting point. Corresponding to such a path, the disk D turns by the specified amount, and then turns back again by the same amount to its starting position. That is to say, if one end of a string were affixed to the rim of D and the other end to a stationary framework, then a motion that corresponds to a closed path in R would be one in which, at the completion of the motion, the string would be back in its original configuration. For, if D makes several full turns in one direction, then the string (which, remember, is confined to the plane in this case) becomes wrapped around D that many times. But D then has to turn back again by the same number of turns in order to complete the motion while the string unwinds itself again and so returns to its starting configuration.

Physically speaking, just as the motions of the ball S corresponding to every closed path in $SU(2)$ can be developed by the continuous exaggerations of a slight wobble, so can the motions of the disk D corresponding to closed paths in R be developed by the continuous exaggerations of a small oscillation of D.

A.4 CYLINDRICAL AND SPHERICAL ROTATION

Hitherto, mathematicians have regarded $O(2)+$ and $O(3)+$ as rotation groups. The concept of "rotation" implies motion so that we are as much concerned with the dynamic process of a body getting from its initial to its final position as we are with the positions themselves. At the beginning of this paper, we described how the movements made by a dog tied by a long lead could be traced by the fact that the rope was wrapped around his kennel. This is a very old idea that deserves to be raised to a principle: *the final configuration of a line connecting a moving point to an observer's stationary framework represents the motion of that point.*

Because, superficially, we discern the universe as being divided into matter and empty space, and because we habitually think of the latter as being the very epitome of nothingness, we are accustomed to thinking of motion as being something absolute—something unimpeded by interrelationships. In the past, mathematicians have described motion simply by giving the beginning and the end of a journey. However, in this paper, we maintain that motion is relative, and that there is, at all times, a topological connection between relatively moving objects. In two-dimensional rotation, therefore, the configuration of the string (the dog's leash) characterizes the rotation of D. In fact, if the string is played out from a fixed point next to the rim of D, if it is kept taut, and if D is of unit radius, then the length of the string becomes the radian measure of the rotation of D. Furthermore, it is positive or negative according to whether it is wound onto D in a counterclockwise or clockwise direction. It is also the measure of the distance from O to the relevant point (designating the rotation) in the group space R. Consequently, we can see that the rotation group in two dimensions is R not $O(2)+$.

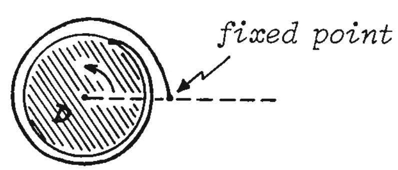

Fig. 23. String winding around a disc rotating in the plane.

Similarly, we have described three-dimensional rotation by attaching one end of a string to a rotating object and the other end to a stationary framework. It is the configuration of the string that characterizes the rotation. We have, in this paper, erected a model in which all the possible configurations of a string are represented by points in the group space $SU(2)$. Thus, *the rotation group in three dimensions is* $SU(2)$, *not* $O(3)+$. And, furthermore, *two full turns of this model are equivalent to not having turned at all!*

This seems to confound our very experience. To explain the anomaly, we shall draw upon an analogous example in geometry. Suppose we consider a circle in a plane, and then make a list of some of its salient properties. For example:

(1) It is the locus of a point that moves so as to be a constant distance from a fixed point called the center.

(2) Its equation in the standard Cartesian coordinate system is

$$x^2+y^2=r^2$$

Now if we ask for the three-dimensional analog of the circle, two possibilities arise: either we may draw the circle out in the third direction perpendicular to the plane so as to form a figure in which we can map every point into a line. In this case the above properties transform into the following:

(1′) It is the locus of a line that moves so as to be a constant distance from a fixed line called the axis.

(2′) Its equation in the standard Cartesian coordinate system is

$$x^2+y^2=r^2$$

Or, we may allow the third dimension to participate symmetrically with the initial two and end up with a figure whose properties are as follows:

(1″) It is the locus of a point that moves so as to be a constant distance from a fixed point called the center.

(2″) Its equation in the standard Cartesian coordinate system is

$$x^2+y^2+z^2=r^2$$

These figures are, of course, respectively, the cylinder and the sphere.

Similarly, we have two-dimensional rotation with its group R and punctiform center. There are, again, two principle extensions of this dynamic figure into three dimensions. The first is what we shall call *cylindrical rotation*. Here, the center of rotation in the plane is drawn out into an axis of rotation, and the mathematical description of it is again the group space R. Thus, angle again becomes the appropriate measure for this type of rotation. It is, furthermore, the only type that has ever been considered in the corpus of classical mechanics.

The second type is what we shall call *spherical rotation*, whose mathematical description is the group space $SU(2)$. In it, the three directions of space participate more symmetrically in the motion.[11] It has not hitherto been recognized or fully understood for the reason that hardly any examples of it occur in the world of our direct physical experiences. All of the rotating things we normally see—spinning tops, motor shafts, wheels, etc.—exhibit rotation of the cylindrical type. To explain why this is, we reemphasize that the string method of classifying rotation incorporates a description of the connection between the object and a stationary framework.

With cylindrical rotation, its group R is infinite or *noncompact* (to use topological terminology). If an object rotates in this mode, the point representing the string configuration moves steadily out along R in one direction to become further removed from its starting position. Physically speaking, the string becomes more and more wound onto the rotating body (or twisted up off the end of it), and so a connecting medium between the body and a stationary framework would have to possess infinite powers of extension. In fact such a medium does not exist, and the only two alternatives for avoiding this confusion characterize cylindrical rotation. The first is that the connecting medium shears and so develops a surface of discontinuity. Such a surface, called a bearing surface, is well known in mechanical examples where there has to be a supporting connection between the rotating body and its stationary reference frame. The other possibility is that the connecting medium holds, but that the stationary framework yields to take up the motion of the rotation. In this case, the rotation is transferred to the connected body. There are, then, these two aspects of cylindrical rotation; namely, the existence of bearing surfaces, and (or) the transmission of the rotation to or from the moving body.

On the other hand, with spherical rotation, the group space $SU(2)$ is *compact*, and the path representing a steady rotation about a fixed axis is closed and finite, so that the string passes through a cyclic configuration to keep returning to its initial position. The interconnecting medium in this case does not shear, nor does the stationary framework have to yield. Thus,

[11]The concept of an "axis" of rotation is now severely localized. It only extends over the core region, that is, the ball S in our above-quoted models. This ball could, theoretically, be contracted to zero radius, leaving only a rotating point and the undulating medium. This would be the strict model of spherical rotation in which the analogy with the geometric sphere would be exact.

spherical rotation is characterized by the following: namely, the rotating body does not transmit or receive rotational energy, but is surrounded by a medium that undergoes a cyclical "wave" motion. This makes the notion of spherical rotation elusive to our senses, because, in the world of macrophysics, motion has generally to be maintained against dissipative forces like friction. But spherical rotation cannot receive energy from a propulsive source. In the example given near the beginning of this paper of a steel ball embedded in jelly, it would be impossible to sustain the motion mechanically, say by a driveshaft or the like, because such a connection would necessarily act in the cylindrical mode. An effective driveshaft would clash with the undulating motions of the jelly and rip it to pieces. For this reason, we suggested that the ball be magnetic and be propelled by magnetic forces. A mechanical model of spherical rotation may still be propelled by the overriding magnetic forces acting in the cylindrical mode. (Note, however, that an electromagnetic example of spherical rotation could not exchange electromagnetic energy.) There are other ways to build visible mechanical models of spherical rotation, but to run them requires some sophisticated device that is essentially a bit of trickery as far as the purity of the model is concerned (see footnote 10).

5A. MATHEMATICAL REPRESENTATION OF SPHERICAL ROTATION

In Section A.1, we stated that the Lie group space $SU(2)$ consisted of two spheres and their interiors. If these two spheres were superimposed, their surface points would be identified in diametrically opposite pairs. By examining an analogous situation in two dimensions, we shall see how to construct a more integrated model of this space.

Consider two disks with their perimeter points identified in what would be diametrically opposite pairs if the two disks were superimposed (Figure 24). If these two disks were embedded in a three-dimensional Euclidean space and deformed into two hemispherical surfaces, they would fit together with identified point-pairs coinciding so as to form a sphere. Thus, the whole space consisting of the two disks becomes equivalent to a complete spherical surface.

Likewise, the two spherical surfaces of $SU(2)$ can be embedded in four-dimensional Euclidean space and deformed into two four-dimensional hyperhemispheres. They can then be brought together over a spherical surface of contact in such a way that all identified point pairs coincide. The result is the bounding volume of a four-dimensional hypersphere. The

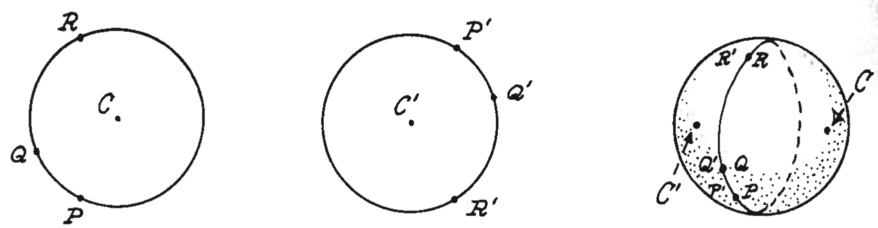

Fig. 24. Two disks, in which the diametrically opposite circumferential points on alternate disks are identified, are together equivalent to the surface of a sphere. The centers C, C' end up diametrically opposite on the sphere.

centers O and O' of the original two components of $SU(2)$ become diametrically opposite points.

How large is this hypersphere? We recall that in the space of $SU(2)$, each sphere had a radius of π units corresponding to the radian measure of half a turn of the physical body being rotated. Thus, a complete circuit of $SU(2)$ corresponds to a double turn of the ball and has a length of 4π units. In a hyperspherical model of $SU(2)$, this circuit corresponds to one of the great circles formed by the intersection of the hypersphere with a plane through the center. It follows that the radius of the hypersphere is 2 units. In switching our attention from the motion of the ball to that of the string configuration (of Figure 16), we note that it rotates at half the frequency of the ball. Therefore, if we use the measure of its phase angle to induce a measure on the hypersphere, the latter will have a radius of one unit. Of key importance is the fact that each point in the bounding volume of the hypersphere represents, uniquely, a particular configuration of the spherical rotation. We lose nothing by declaring that the radius of the hypersphere can be chosen to be any length.

SUMMARY

Just as the circle in the plane relates to both the cylinder and the sphere, so, too, does rotation in the plane relate to the cylindrical and spherical modes of three-dimensional rotation. Classical mechanics has, hitherto, been deficient in that it has only recognized the cylindrical mode. Spherical rotation is the simplest mode in which one part of space can spin in relation to another without disrupting its continuity. A moving vortex spinning in the spherical mode satisfies Dirac's equation for an elementary particle. Hence, mass and energy can be explained as being manifestations of the rotation of space. Our geometric model demonstrates the difference

not only between spin-up and spin-down states, but also between the particle and its antiparticle.

The spherical rotation of space has a spinning center that can be identified with the position of the particle. As such, hidden variables exist; that is to say, there always exist an exact particle location and, generally, an exact momentum. It is an intrinsic feature of spherical rotation that the spin cannot be transmitted to or from the core, and, therefore, the position of the particle can be approximated only by bracketing the whole region of the disturbance—a procedure that also renders the momentum inexact. In this way do measurements become circumscribed by the indeterminacy principle.

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