Calculating Synergetics' Tetra-volumes of Polyhedra

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To calculate the volume of an object you need a unit reference volume and a unit reference length. The unit reference volume has typically been a Cube shape, but you could use a sphere or any other shape, including an irregular shape. In Synergetics, the regular Tetrahedron is used as the unit reference volume shape.

The unit reference length is chosen to be some linear measure or feature of the unit reference volume. For example, if the sphere is the unit reference volume then the sphere's radius might be chosen as the unit reference length, or the sphere's diameter might be chosen. For the standard Cube system the Cube's edge is chosen. For the regular Tetrahedron measurement system of Synergetics the edge of the Tetrahedron is chosen as the unit reference length.

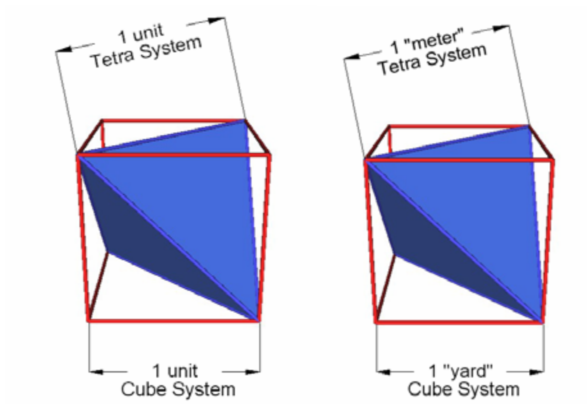

Figure 1 Cube and Tetrahedron system measurement.

The volume equations for the unit reference volumes and unit reference lengths are then written as

Cube system: $V_C(EL_C) = EL_C^3$

Tetra system: $V_T(EL_T) = EL_T^3$

Note that in Synergetics the unit edge length of the Cube is not the same physical length as the unit edge length of the Tetrahedron. One way to think of this is to consider the difference between 1 yard and 1 meter. Both are units of measure (with a length of 1 unit) but they are physically not the same length. There is a conversion factor between the two length measurement systems.

In the case of Synergetics, the edge of the unit volume regular Tetrahedron is associated with a square face diagonal of the unit volume Cube. So the length conversion equations are given by

$$L_T = \frac{1}{\sqrt{2}} L_C \qquad (1)$$

That is, when the Cube system's "yard" is used to measure the reference length of the Tetra system, which is the face diagonal of the unit Cube, the Cube system "yard" will measure $\sqrt{2}$ Cube "yards". But this is defined to be one Tetra system's "meter". Hence the need to divide the Cube system's "yard" measurement by $\sqrt{2}$ to get the Tetra system's "meter" measurement.

$$L_C = \sqrt{2}\, L_T \qquad (2)$$

When the Tetra system's "meter" is used to measure the Cube system's reference length, which is the edge of the Cube, the Tetra system's "meter" will measure $\frac{1}{\sqrt{2}}$ Tetra "meters". But this is defined to be one Cube system's "yard". Hence the need to multiple the Tetra system's "meter" measurement by $\sqrt{2}$ to get the Cube system's "yard" measurement.

The next problem is to relate the two different volume measurement systems for an arbitrary object. That is, not only do we need conversion factors between the linear measurement systems, we also need a conversion factor between the volume measurement systems.

Using the observation that 3 unit volume Tetrahedra as defined in the Tetra "meter" system fills 1 unit volume Cube as defined in the Cube "yard" system, we have

$$(1 \text{ unit length "yard" in Cubic system})^3 = 3\,(1 \text{ unit length "meter" in Tetra system})^3 \qquad (3)$$

where, as described above,

$$1 \text{ unit length "yard" in Cubic system} \neq 1 \text{ unit length "meter" in Tetra system}$$

We can now define the steps for deriving the volume equations that Synergetics uses for various polyhedra. This is done for a few polyhedra in the following examples.

Example using a Cube:

1) Pick a linear component or feature of the object as the Cube system unit linear reference feature to measure.

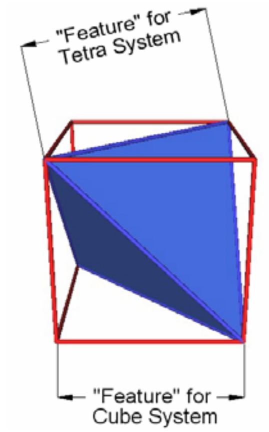

Figure 2 Selecting "feature" that is to be associated with the Cube system unit of length.

The standard Cube system of measure uses the Cube's edge as the feature for the unit of measure.

EL_C = Edge length of the Cube.

2) Calculate the Cube based system's volume equation based on that linear feature.

$V_C = 1\ EL_C^3$

3) If this is not the same unit linear feature to be used by the Tetra measurement system then convert the Cube system volume equation to use the same unit linear feature to be used in the Tetra-based measurement system.

a) In this case the Cube's face diagonal (FD) is to be used as the linear feature in the Tetra system. In the Cube measurement system, these linear features are related by the equation

$$EL_C = \frac{1}{\sqrt{2}}\, L_{C\text{-}FD}$$

b) Convert the Cube system volume equation to use the linear feature length to be used in the Tetra system, but still as measured in the Cube system.

$$V_C = 1\ EL_C^3 = 1\ \left(\frac{1}{\sqrt{2}}\, L_{C\text{-}FD}\right)^3 = \frac{1}{2\sqrt{2}}\, L^3_{C\text{-}FD}$$

4) Convert from the Cube linear measure (“yard”) to the Tetrahedron based linear measure (“meter”) and multiply by the Cube to Tetrahedron volume conversion factor.

V_T = VolumeConversionFactor × V_C(L_C = LengthConversionFactor × L_T)

$$V_T = 3 \times \frac{1}{2\sqrt{2}} \left(\sqrt{2}\ EL_T\right)^3$$

$$V_T = 3\, EL_T^3$$

This is the Synergetics Tetra-volume equation for the Cube.

Example using the Octahedron:

1) Pick a linear component or feature of the object to associate with the Cube system's linear measure (which is the Cube's edge).

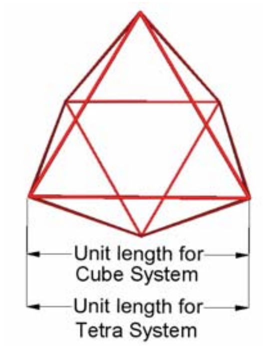

Figure 3 Octahedron and "feature" for defining the unit linear measure in the Cube and the Tetra systems.

$EL_O = EL_C$

(Edge length of Octahedron is to be associated with the edge length of the Cube for calculating the Cube system based volume equation.)

2) Calculate the Cube based system's volume equation based on that linear feature.

Consider the Octahedron as 2 square based pyramids. The Cubic system volume equation is given by

$V_P = (1/3)(\text{base area})(\text{height})$

base area = EL_O^2

height = Distance from Octahedron's center of volume to a vertex.

$$= DVV = \frac{1}{\sqrt{2}} EL_O$$

$$V_P = \frac{1}{3}\left(EL_O^2\right)\left(\frac{1}{\sqrt{2}} EL_O\right) = \frac{1}{3\sqrt{2}} EL_O^3$$

So the cube system volume equation for the Octahedron is

$$V_{O\text{-}C} = \frac{2}{3\sqrt{2}} EL_{O\text{-}C}^3$$

3) If this is not the same unit linear feature to be used by the Tetrahedron measurement system then convert the above Cube system volume equation to use the same unit linear feature to be used in the Tetra-based measurement system.

> It turns out that Synergetics also uses the Octahedron's edge as the feature to define the Tetra system's unit linear measure so there is nothing to do for this step.

4) Convert from Cube linear measure ("yard") to Tetrahedron based linear measure ("meter") and multiply by the Cube to Tetrahedron volume conversion factor.

$V_{O\text{-}T}$ = VolumeConversionFactor × $V_{O\text{-}C}$($EL_{O\text{-}C}$ = LengthConversionFactor × L_T)

$$V_{O\text{-}T} = 3 \times \frac{2}{3\sqrt{2}} \left(\sqrt{2}\ EL_T\right)^3 = 3 \times \frac{2 \times 2\sqrt{2}}{3\sqrt{2}}\ EL_T^3$$

$$V_{O\text{-}T} = 4\ EL_T^3$$

This is the Synergetics Tetra volume equation for the Octahedron.

Example using the A Quantum Module:

1) Pick a linear component or feature of the object to associate with the Cube system's unit linear measure (the Cube's edge).

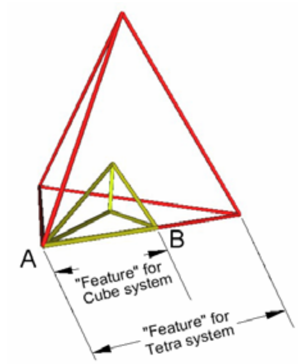

Figure 4 Tetrahedron and AQM (yellow) and the "feature" selected for unit linear measure in the Cube and Tetra systems.

I'll pick what I call the AB-edge of the AQM. This edge is also part of the Tetrahedron's edge by the way that the AQM is defined within the Tetrahedron.

$EL_{AB\text{-}AQM\text{-}C} = EL_C$

2) Calculate the Cube based system's volume equation based on that linear feature.

Consider the AQM as a triangular based pyramid. The Cubic system volume equation is given by

V_P = (1/3)(base triangle area)(pyramid height)

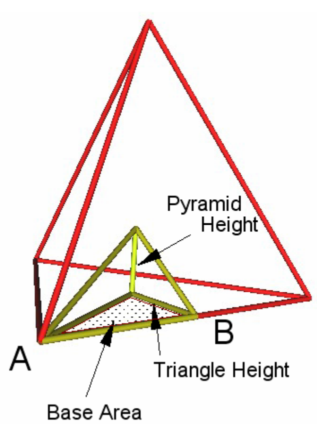

Figure 5 "Base Area" and "Height" used in defining the Cube system volume equation.

base triangle area = (1/2)($EL_{AB\text{-}AQM\text{-}C}$)(triangle-height)

For the AQM, triangle-height = $\frac{1}{\sqrt{3}} EL_{AB\text{-}AQM\text{-}C}$

pyramid height = $\frac{1}{\sqrt{6}} EL_{AB\text{-}AQM\text{-}C}$

$$V_{AQM\text{-}C} = \frac{1}{3}\left(\frac{1}{2} EL_{AB\text{-}AQM\text{-}C}\right)\left(\frac{1}{\sqrt{3}} EL_{AB\text{-}AQM\text{-}C}\right)\left(\frac{1}{\sqrt{6}} EL_{AB\text{-}AQM\text{-}C}\right)$$

$$V_{AQM\text{-}C} = \frac{1}{18\sqrt{2}} EL^3_{AB\text{-}AQM\text{-}C}$$

3) If this is not the same unit linear feature to be used by the Tetrahedron measurement system then convert the volume equation to use the same unit linear feature to be used in the Tetra-based measurement system.

a) We have taken the AB edge length to be the unit linear measure “yard” in the Cube system. But the feature which is to become the unit linear measure “meter” in the Tetra system is the edge length of the Tetrahedron, which is twice the length of the AB edge of the AQM. So $EL_{AB\text{-}AQM\text{-}C}$ that we have been using in the Cube system is ½ the length of the feature that will be used in the Tetra system.

$$EL_{AB\text{-}AQM\text{-}C} = \frac{1}{2} L_C$$

b) Convert the Cube system’s volume equation to use the linear feature that is to become the unit linear measure in the Tetra system. Note that this is still as measured in the Cube system (“yard”).

$$V_{AQM\text{-}C} = \frac{1}{18\sqrt{2}}\left(\frac{1}{2} L_C\right)^3 = \frac{1}{144\sqrt{2}} L^3_C$$

4) Convert from Cube linear measure ("yard") to Tetrahedron based linear measure ("meter") and multiply by the Cube to Tetrahedron volume conversion factor.

$$V_{\text{AQM-T}} = \text{VolumeConversionFactor} \times V_{\text{AQM-C}}(L_C = \text{LengthConversionFactor} \times L_T)$$

$$V_{\text{AQM-T}} = 3 \times \frac{1}{144\sqrt{2}} \left(\sqrt{2}\ EL_T\right)^3$$

$$V_{\text{AQM-T}} = 3 \times \frac{2\sqrt{2}}{144\sqrt{2}}\ EL_T^3$$

$$V_{\text{AQM-T}} = \frac{1}{24}\ EL_T^3$$

This is the Tetra-system volume equation for the A Quantum Module.

Note that this is in terms of the edge length of the regular Tetrahedron and ***not*** in terms of the AQM's AB edge length.

Example using the E Quantum Module:

$$\varphi = \frac{1+\sqrt{5}}{2}$$

1) Pick a linear component or feature of the object to associate with the Cubic-system's linear measure.

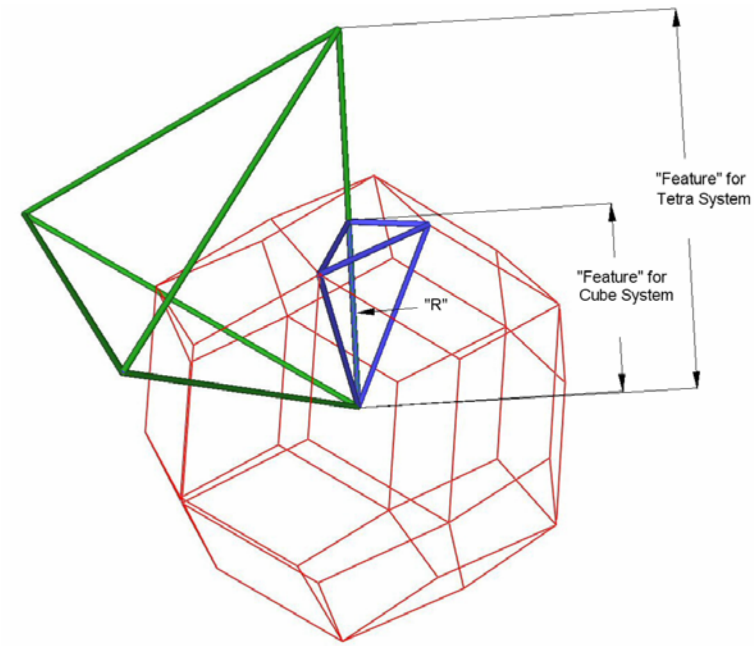

Figure 6 E Quantum Module (blue) in rhombic Triacontahedron (red) and the "feature" for Cube system ("R") and the "feature" for the Tetra system (green Tetrahedron edge).

I'll pick the distance from the rhombic Triacontahedron's center of volume to the center of a diamond's face, which I'll label as "R".

$R_{EQM-C} = EL_C$

2) Calculate the Cube based system's volume equation based on that linear feature.

Consider the EQM as a triangular based pyramid. The Cubic system volume equation for such a pyramid is given by

V_P = (1/3)(base triangle area)(pyramid height)

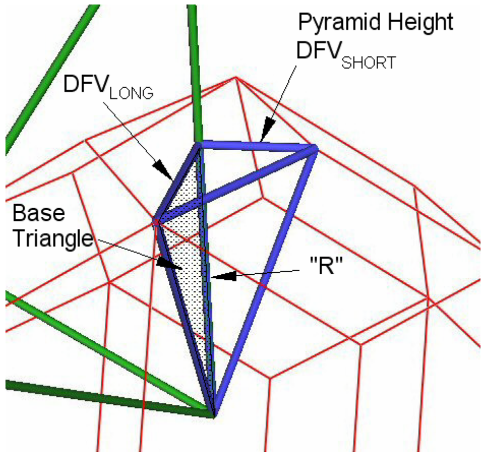

Figure 7 Labeling parts of the EQM.

base triangle area = $(1/2)(R_{EQM\text{-}C})$(triangle-height)

For the EQM,

triangle-height = Distance from face center to long vertex

$$= DFV_{LONG} = \frac{1}{\varphi} R_{EQM\text{-}C}$$

pyramid height = Distance from face center to short vertex

$$= DFV_{SHORT} = \frac{1}{\varphi^2} R_{EQM\text{-}C}$$

where "long vertex" means the face vertex having an opposite face vertex along the long face diagonal and
where "short vertex" means the face vertex having an opposite face vertex along the short face diagonal.

This gives

$$V_{EQM\text{-}C} = \frac{1}{3}\left(\frac{1}{2} R_{EQM\text{-}C}\right)\left(\frac{1}{\varphi} R_{EQM\text{-}C}\right)\left(\frac{1}{\varphi^2} R_{EQM\text{-}C}\right)$$

$$V_{EQM\text{-}C} = \frac{1}{6}\,\frac{1}{\varphi^3}\, R^3_{EQM\text{-}C}$$

3) If this is not the same unit linear feature to be used by the Tetrahedron measurement system then convert the volume equation to use the same unit linear feature to be used in the Tetra-based measurement system.

a) We have taken the length “R” to be the unit linear measure “yard” in the Cube system. But the feature which is to become the unit linear measure “meter” in the Tetra system is the edge length of the Tetrahedron, which is twice the length of “R”. So the “R” that we have been using in the Cube system is ½ the length of the feature that will be used in the Tetra system as the Tetra system unit of linear measure.

$$R_{\text{EQM-C}} = \frac{1}{2} L_C$$

where L_C is the edge length of the Tetrahedron as measured by the Cube system unit of measure “yard”.

b) Convert the Cube system’s volume equation to use the linear feature that is to become the unit linear measure in the Tetra system. Note that this is still as measured in the Cube system (“yard”).

$$V_{\text{EQM-C}} = \frac{1}{6} \frac{1}{\varphi^3} \left(\frac{1}{2} L_C \right)^3$$

$$V_{\text{EQM-C}} = \frac{1}{48} \frac{1}{\varphi^3} L_C^3$$

4) Convert from Cube linear measure (“yard”) to Tetrahedron based linear measure (“meter”) and multiply by the Cube to Tetrahedron volume conversion factor.

$$V_{\text{EQM-T}} = \text{VolumeConversionFactor} \times V_{\text{EQM-C}}(L_C = \text{LengthConversionFactor} \times L_T)$$

$$V_{\text{EQM-T}} = 3 \times \frac{1}{48} \frac{1}{\varphi^3} \left(\sqrt{2}\, EL_T \right)^3$$

$$V_{\text{EQM-T}} = \frac{\sqrt{2}}{8\,\varphi^3} EL_T^3 \simeq 0.04173132\, EL_T^3$$

This is the Synergetics Tetra system volume equation for the E Quantum Module.

Note that this is in terms of the edge length of the regular Tetrahedron and ***not*** in terms of any of the E Quantum Module’s edge lengths.