The Jitterbug Motion

By

Robert W. Gray
180-4 Poplar St.
Rochester, NY 14620
rwgray@rwgrayprojects.com

Draft: 09-24-2002

Introduction

We develop a set of equations which describes the motion of a vertex of the Jitterbug. The Jitterbug starts in the "closed" Octahedron position with 8 triangle faces and 6 vertices and "opens" into a Cuboctahedron (also called the Vector Equilibrium or VE) with 8 triangle faces, 6 square faces, and 12 vertices. (See Figure #1.)

The Jitterbug motion is visually complex but simple when you focus only on the motion of one of the 8 triangles. The motion of a triangle is simply a radial displacement plus a rotation around the radial displacement vector. Because the triangular faces do not change size as they move radially and rotate, the 3 vertices of a triangle are always on the surface of a cylinder. (See Figure #2.) The cylinder is axially aligned with the displacement vector. That is, with a line passing through the Octahedron's (and VE's) center of volume out through the triangle's face center point. There are 4 axes of rotation (two opposite triangular faces per axis) so there are 4 fixed cylinders in which the triangles move.

The reason the motion appears complex is because the Jitterbug is often demonstrated by holding the "top" and "bottom" triangle fixed while pumping the model. This causes the remaining 6 triangles to move radially in and out, rotate *and* orbit about the "up" and "down" pumping axis. By allowing all 8 triangles to move in the same way, that is, not fixing any of the 8 triangles, the motion is simplified.

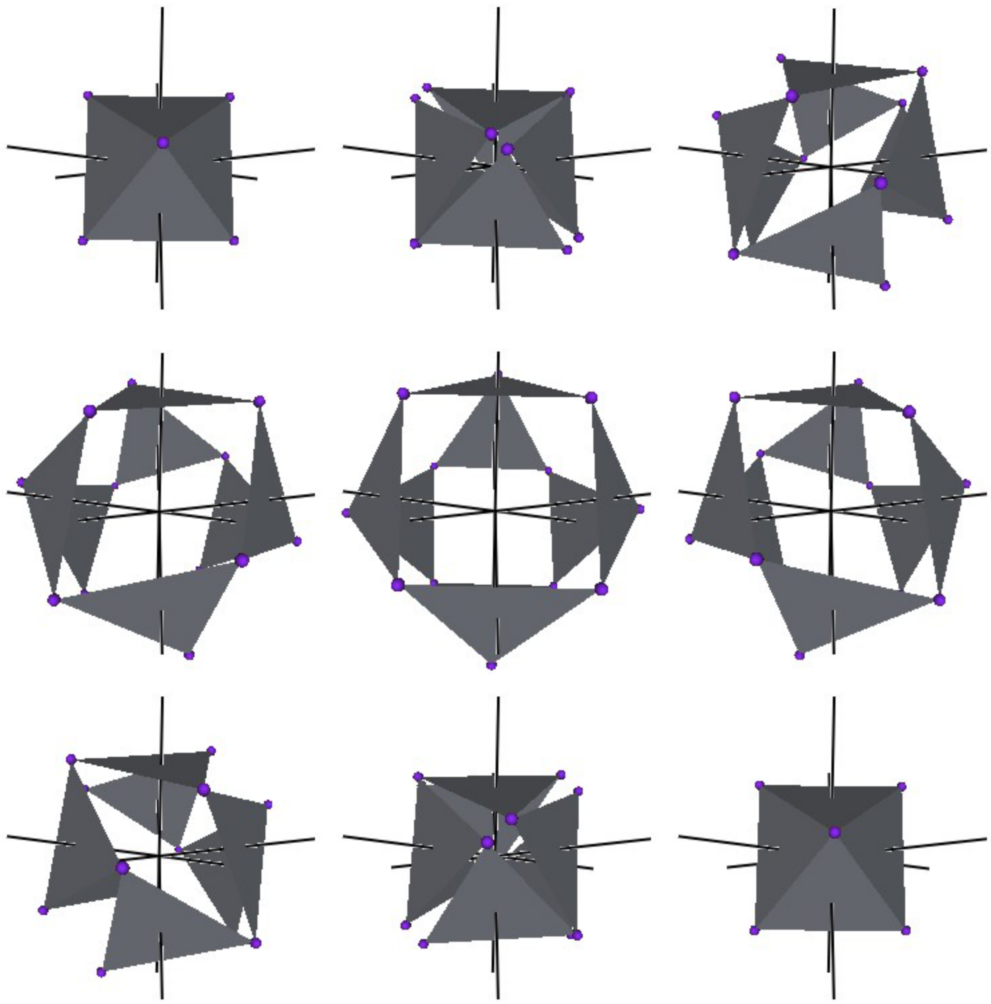

Figure #1 *Jitterbug Motion*

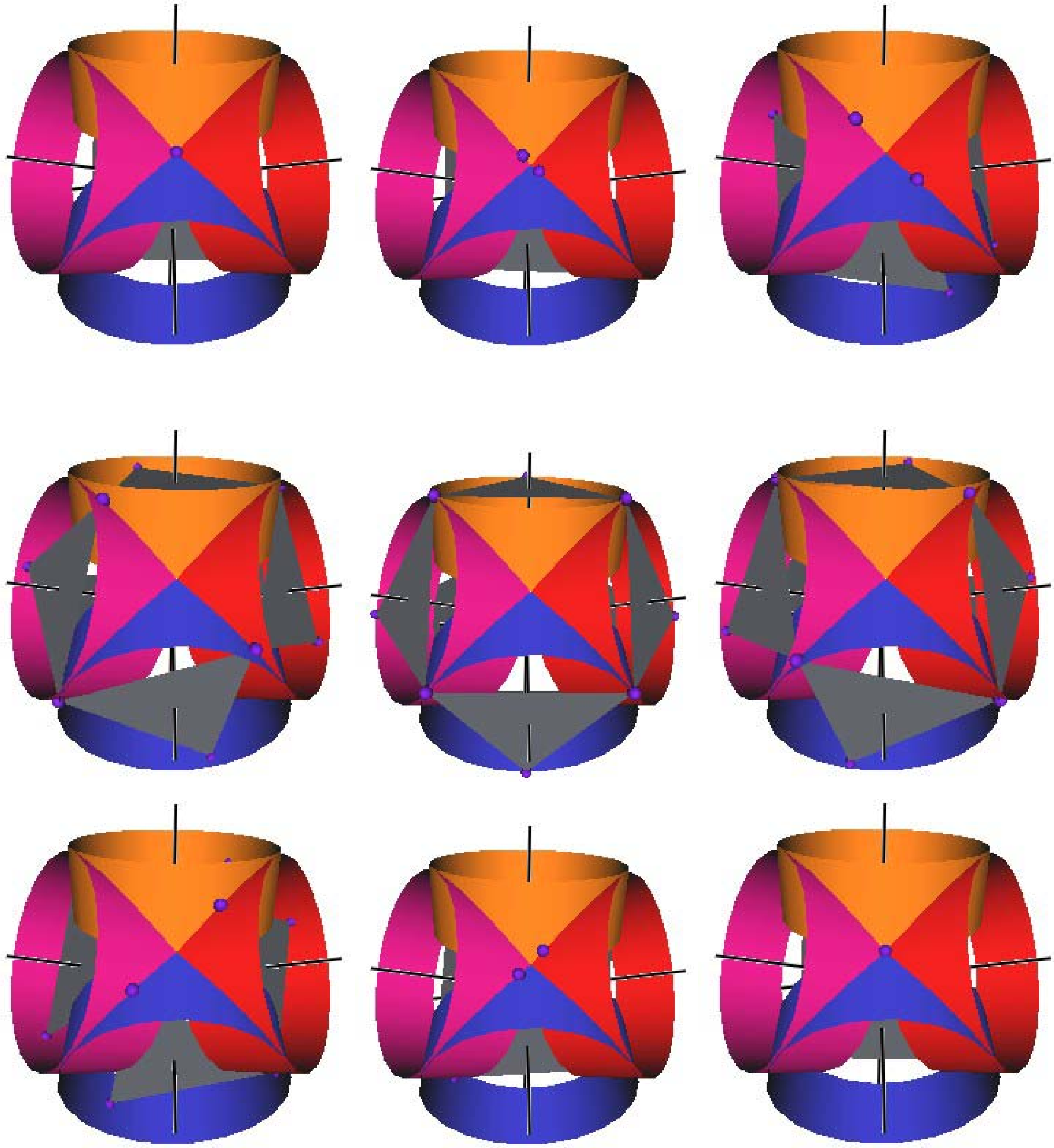

Figure #2 *Jitterbug Motion within cylinders.*

Development of Vertex Motion Equations

Consider the Octahedron so positioned as to have one of its triangular faces in the xy-plane and so that its face center is at the coordinate origin. One of the triangle's vertices is on the –x-axis (minus x-axis).

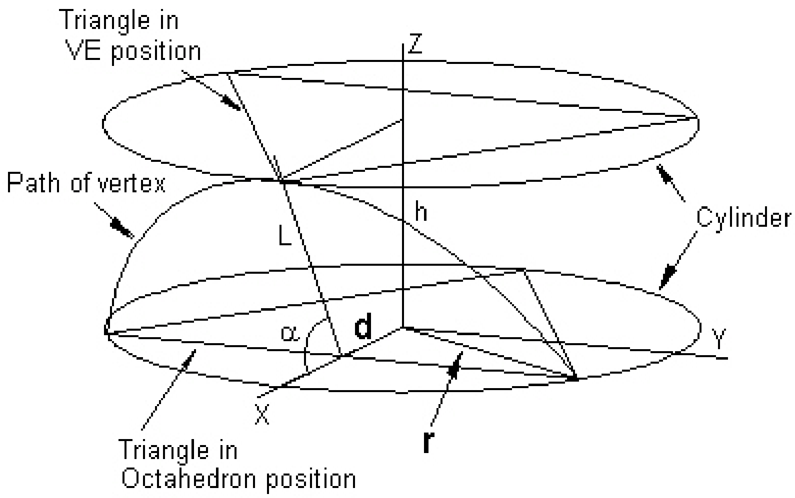

Figure #3 *Path of a vertex*

Let

EL = Edge Length of a triangle face

r = Cylinder radius.

Note that r = DFV = Distance from a triangle's face center to the triangle's vertex.

$$r = \frac{1}{\sqrt{3}}\ EL \qquad \text{and so} \qquad EL = \sqrt{3}\ r$$

The equations for the cylinder in which the triangle face moves is given by

$x_c = r\cos(\varphi)$

$y_c = r\sin(\varphi)$

z_c

During the Jitterbug motion from an Octahedron position through the VE position to the second Octahedron position, a triangular face will rotate from $-60° \leq \varphi \leq 60°$. In other words, it will rotate through an angular range of 120.

The total distance through which the triangular face will move radially from the Octahedron position to the VE position can be calculated as follows:

$$h = T.A. - DVF_o$$

where T.A. is the Tetrahedron Altitude and DVF_o is the distance from the center of volume to the Octahedron's triangle face center point.

$$h = \frac{\sqrt{2}}{\sqrt{3}}\ EL - \frac{1}{\sqrt{6}}\ EL$$

(We have set the edge length of the Tetrahedron to be the same as the edge length of the Octahedron.)

So

$$h = \frac{1}{\sqrt{6}}\ EL \cong 0.40824829\ EL$$

$$h = \frac{1}{\sqrt{2}}\ r \cong 0.707106781\ r$$

The equation for the plane passing through the cylinder is given by

$$x_p = \frac{1}{\sqrt{2}}\ z + (1/2)\ r$$

with $0 \leq z \leq \frac{1}{\sqrt{2}}\ r$ (or $0 \leq z \leq \frac{1}{\sqrt{6}}\ EL$)

Setting $x_c = x_p$, the intersection of the plane and the cylinder, we get

$$r\cos(\varphi) = \frac{1}{\sqrt{2}}\ z + (1/2)\ r$$

or

$$z = \sqrt{2}\ (\cos(\varphi) - 1/2)\ r$$

In terms of EL, we have

$$z = \frac{\sqrt{2}}{\sqrt{3}} \; (\cos(\varphi) \; - 1/2) \text{ EL } \cong \; 0.81649658 \; (\cos(\varphi) \; - 0.5) \text{ EL}$$

This equation relates the radial distance that the triangle moves given the angular amount that it rotates. Note that for this equation, φ = 60° is the closed Octahedron position, φ = 0° is the fully expanded VE position, V = – 60° is the second closed Octahedron position.

We simple invert this equation if we want to know the angular amount the triangle face rotates given the radial distance z that it moves. (Of course, EL > 0.)

$$\cos(\varphi) = \frac{\sqrt{3}}{\sqrt{2} \text{ EL}} \; z \; + 1/2 \; \cong (1.224744871 \text{ / EL}) \; z + 0.5$$

Recall that $0 \leq z \leq \frac{1}{\sqrt{6}}$ EL and that cos(φ) = cos(– φ) so that there are 2 possible angles (φ and – φ) for each z value.

We now develop equations which will allow us to draw the curve which a vertex of the Jitterbug traces during its Jitterbug motion.

From Figure #3,

$$\tan(\alpha) = h/d = h \text{ / } [(1/2) \text{ r}] = \frac{1}{\sqrt{2}} \; r \text{ / } [(1/2) \text{ r}] = \sqrt{2}$$

This implies that α = 54.73561032…°. In terms of sines and cosines, we can write

$$\sin(\alpha) = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\cos(\alpha) = \frac{1}{\sqrt{3}}$$

Also from Figure #3 we see that

$$L \sin(\alpha) = h$$

Or, using z instead of h

$$L = \frac{\sqrt{3}}{\sqrt{2}} z$$

and

$$z = \frac{\sqrt{2}}{\sqrt{3}} L$$

Recall the equation above

$$x_p = \frac{1}{\sqrt{2}} z + (1/2) r$$

and that $x_p = x_c$. Also note that

$$x_c^2 + y_c^2 = r^2$$

so that

$$x_c = \sqrt{r^2 - y_c^2}$$

We then have

$$\sqrt{r^2 - y_c^2} = \frac{1}{\sqrt{2}} z + (1/2) r$$

$$\sqrt{r^2 - y_c^2} = \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{3}} L + (1/2) r$$

$$L = \sqrt{3} \left(\sqrt{r^2 - y_c^2} - (1/2) r \right)$$

Or

$$L = \sqrt{3} \left(\sqrt{\frac{1}{3} EL^2 - y_c^2} - \frac{1}{2\sqrt{3}} EL \right)$$

$$L = \sqrt{EL^2 - 3y_c^2} - \frac{1}{2} EL$$

Relabeling $Z = L$, $Y = y_c$ gives

$$Z = \sqrt{EL^2 - 3Y^2} - \frac{1}{2} EL$$

This equation allows us to plot the path that a vertex travels. (Note that "Z" is not the z-axis used above. This Z is in the plane defined by the vertex motion.)

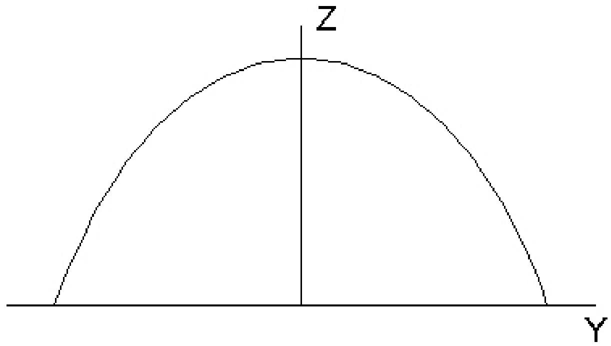

Figure #4 *Jitterbug vertex path*

The ranges are $(-1/2)\ EL \le Y \le (1/2)\ EL$, and $0 \le Z \le (1/2)\ EL$

This curve is part of an ellipse.

The general formula for an ellipse is

$$\frac{(Y - Y_0)^2}{b^2} + \frac{(Z - Z_0)^2}{a^2} = 1$$

where the center of the ellipse is (Y_0, Z_0) and a = semimajor and b = semiminor axis.

Starting with

$$Z = \sqrt{EL^2 - 3Y^2} - \frac{1}{2}\ EL$$

we can write

$$Z + (1/2)\ EL = \sqrt{EL^2 - 3Y^2}$$

$$(Z + (1/2)\ EL)^2 = EL^2 - 3Y^2$$

$$3Y^2 + (Z + (1/2)\ EL)^2 = EL^2$$

$$\frac{Y^2}{\frac{1}{3}EL^2} + \frac{\left(Z + \frac{1}{2}EL\right)^2}{EL^2} = 1$$

This tells us that:

the center of the ellipse, in (Y, Z) coordinates, is at (0, – (1/2)EL),

the semimajor axis is a = EL,

the semiminor axes is b = $\frac{1}{\sqrt{3}}$ EL.

This complete ellipse is shown in the next Figure.

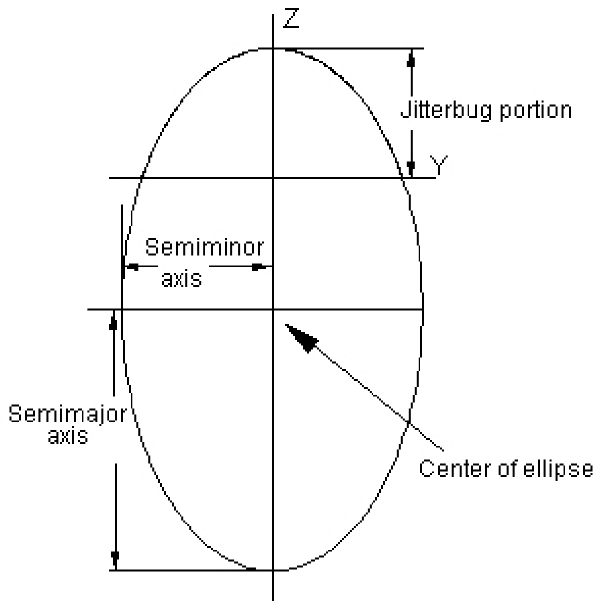

Figure #5 *Jitterbug Ellipse*

From a mathematical point of view, we can change the center of the ellipse without changing its properties. So we change the ellipse equation from

$$\frac{Y^2}{\frac{1}{3}EL^2} + \frac{\left(Z + \frac{1}{2}EL\right)^2}{EL^2} = 1$$

to

$$\frac{Y^2}{\frac{1}{3}EL^2} + \frac{Z^2}{EL^2} = 1$$

which moves the center of the ellipse to (Y, Z) = (0, 0).

The parametric form of the equation for the ellipse is given by

$$Y = \frac{1}{\sqrt{3}} \text{ EL } \sin(\theta)$$

$$Z = \text{EL } \cos(\theta)$$

The eccentricity of an ellipse is defined by the equation

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

Using the above values for a and b:

$a = EL$

$$b = \frac{1}{\sqrt{3}} \text{ EL}$$

we get

$b^2 / a^2 = (1/3) / 1 = 1/3$

so that

$$e = \sqrt{1 - \frac{1}{3}} = \frac{\sqrt{2}}{\sqrt{3}} \cong 0.81649658$$

The coordinates for the 2 focus points are (0, ae) and (0, –ae) in the (Y, Z) plane. These evaluate to

$$(0, \frac{\sqrt{2}}{\sqrt{3}} \text{ EL}) \cong (0.0, 0.81649658 \text{ EL}) \text{ and } (0, -\frac{\sqrt{2}}{\sqrt{3}} \text{ EL}) \cong (0.0, -0.81649658 \text{ EL})$$

Of course, knowing that two intersecting cylinders produce an ellipse and knowing that the Jitterbug's triangles move on the surface of cylinders tells us that the path of the vertices trace out part of an ellipse.

The Jitterbug Ellipses

In Figure #6, the "Jitterbug portion" is the actual path that vertices will travel (direction of travel is not considered here.) No vertex of the Jitterbug (when considering only the Octahedron to VE to Octahedron motion) traverses that portion of the ellipse curve which is within the "Square cross section of Octahedron" portion of the ellipse. Later in this paper we will consider what happens if the vertices are allowed to move along this section of the ellipse.

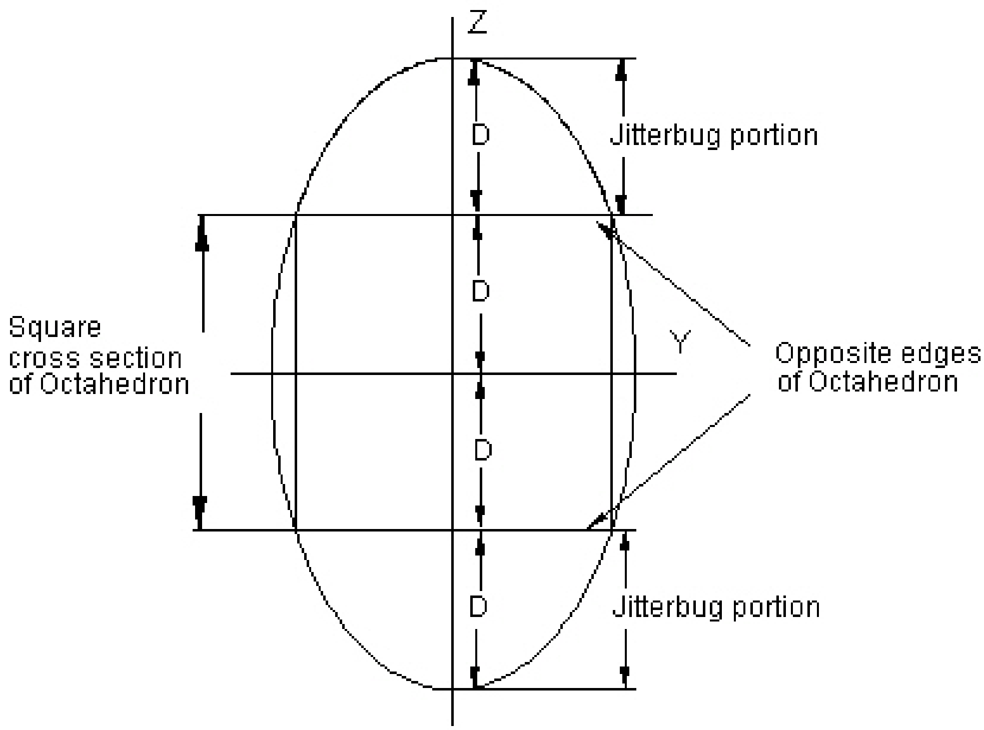

Figure #6 *Ellipse and Octahedron edges*

Note that all four of the square's edges in the ellipse of Figure #6 are Octahedron edges. Each pair of opposite edges of the Octahedron is part of an ellipse. Therefore, there are two orthogonal ellipses in the same plane. Figure #7 shows both ellipses defined by the motion of 4 Jitterbug vertices.

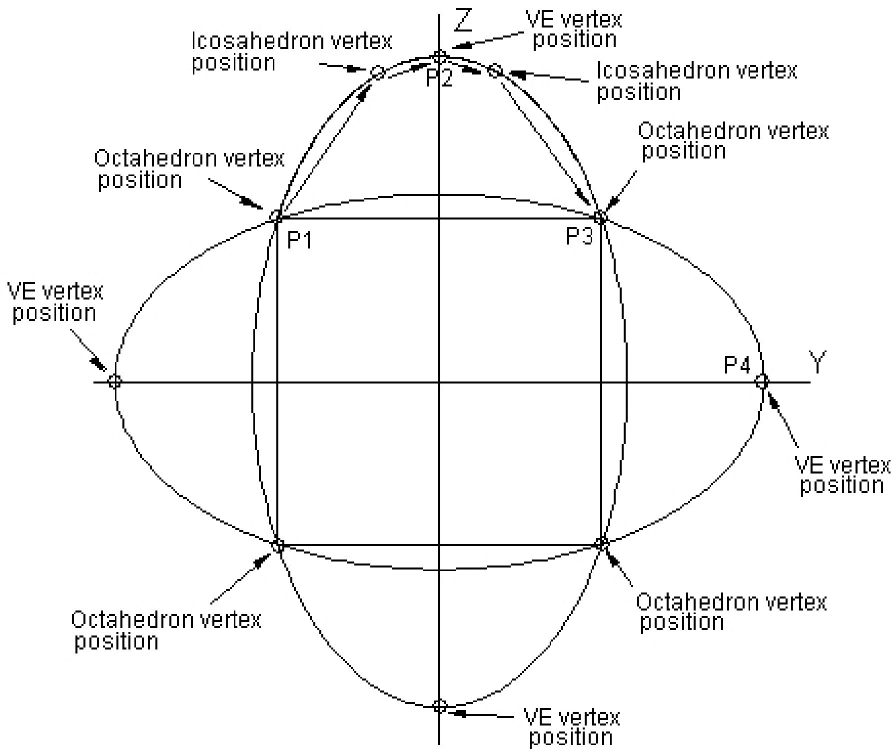

Figure #7 *Two ellipse per plane*

Following only one vertex (one vertex of a rotating Jitterbug triangle) and with the Jitterbug in the Octahedron position, we label the initial vertex position "P1" . This vertex will travel along the ellipse, passing through an Icosahedron position, to reach vertex position "P2", the VE vertex position. Then, with the Jitterbug triangle continuing to rotate in the same direction, the Jitterbug vertex passes through another Icosahedron vertex position to reach the Octahedron position "P3". (Further details relating the Jitterbug vertex position along the ellipse and various polyhedra is given below.) Note that if the Jitterbug triangle were allowed to continue to rotate in the same direction then the vertex now at vertex position "P3" would ***not*** proceed to vertex position "P4". Instead, is leaves this plane to follow another ellipse.

The Octahedron has 12 edges forming 6 opposite edge pairs. So there are a total of 6 ellipses to define the complete Jitterbug motion. These 6 ellipses are show in Figure #8 and Figure #9.

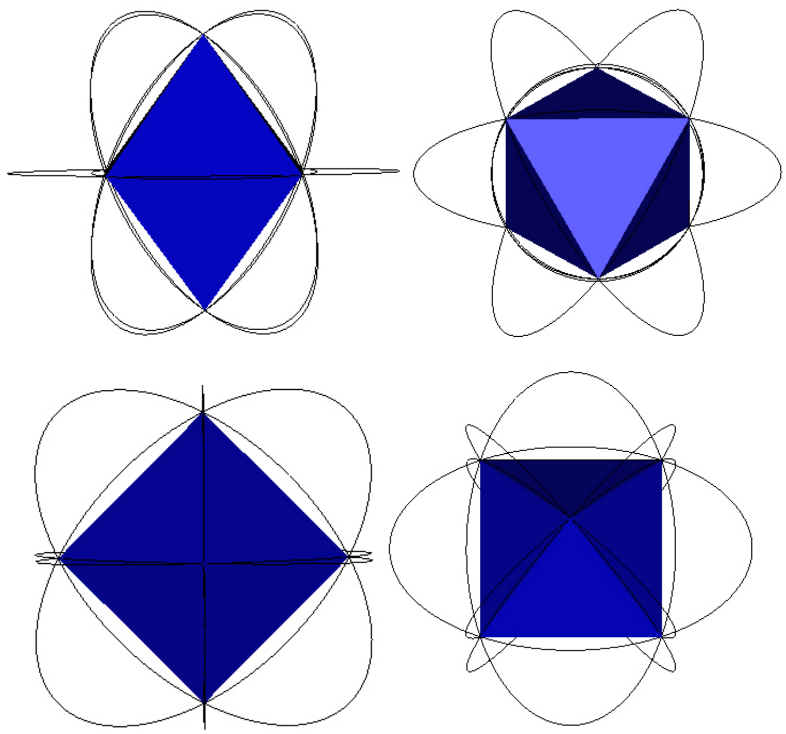

Figure #8 *Six ellipses and Octahedron*

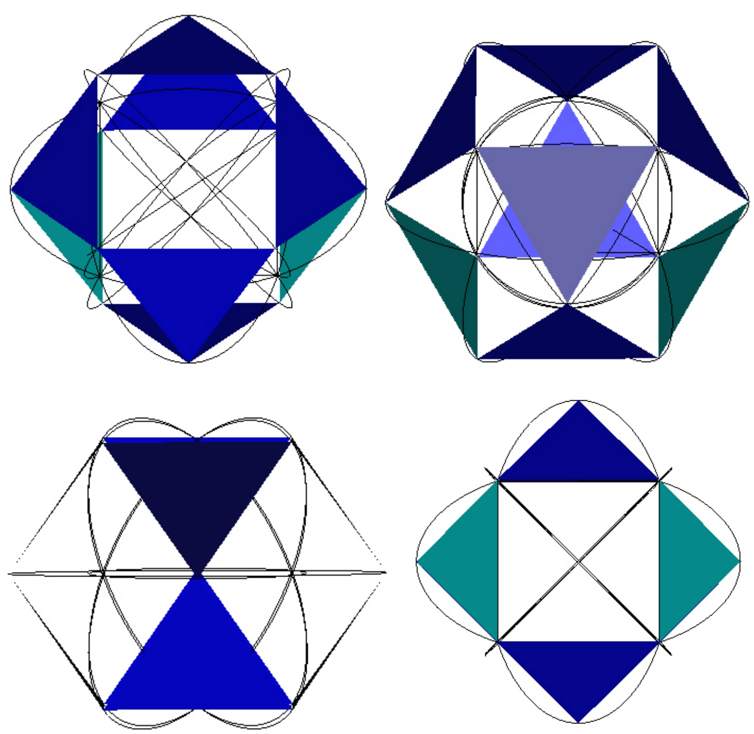

Figure #9 *Six ellipses and the VE*

It is well known, and as mentioned above, that the Jitterbug vertices pass through an Icosahedron position during its Jitterbug motion. (See Figure #11.) What is not well know is that the Jitterbug vertices also pass through a regular Dodecahedron position along the ellipses. (See Figure #12.)

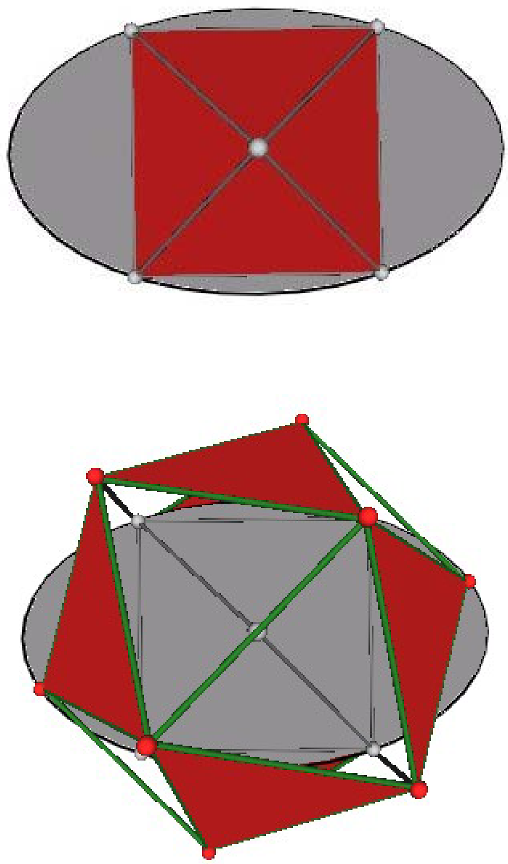

Figure #10 *Jitterbug in Icosahedron position*

Unlike the Jitterbug in the Icosahedron position, not all the vertices of the regular Dodecahedron are defined by one Jitterbug. The Dodecahedron has 20 vertices. The Jitterbug in the Dodecahedron position (as well as in the Icosahedron position) has only 12 vertices. To completely define all 20 vertices of the Dodecahedron *in a symmetrical way* requires 5 Jitterbugs. This gives a total of $5 \times 12 = 60$ vertices. When this is done, the pentagon faces of the Dodecahedron become pentagrams. (It is possible to cover all the Dodecahedron vertices with 3 Jitterbugs but not in a symmetrical way. That is, not in a way as to have each of the Dodecahedron's vertices covered by the same number of Jitterbug vertices and each of the Dodecahedron's faces containing the same number of Jitterbug triangle edges.)

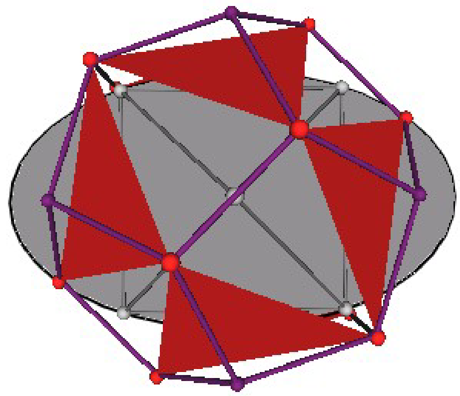

Figure #11 *Jitterbug in regular Dodecahedron position*

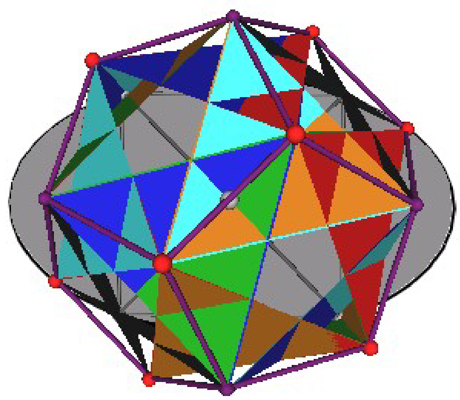

Figure #12 *Symmetrical covering of Dodecahedron by 5 Jitterbugs*

(These 5 Jitterbugs are the basis for the 120 Polyhedron as explained in the paper "What's in this Polyhedron?" which can be found at http://www.rwgrayprojects.com/Lynn/NCH/whatpoly.html)

The Jitterbug in the VE position is shown in Figure #13.

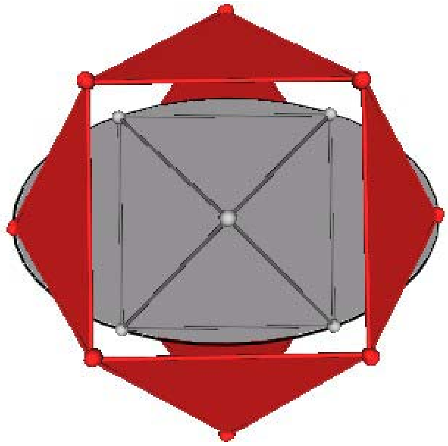

Figure #13 *Jitterbug in the VE position*

Because of the symmetry of the elliptical path, a Jitterbug vertex will pass through 2 Icosahedra and 2 regular Dodecahedra positions. These are shown in Figure #14. The vertex positions labeled “D,C,T” stand for the “Dodecahedron, Cube, Tetrahedron” position. (It is well known that 5 Cubes and 10 Tetrahedra share the same vertices as a regular Dodecahedron.) The positions labeled “O” are the Octahedron positions, those labeled “I” are the Icosahedron positions, and those labeled “VE” are the VE positions.

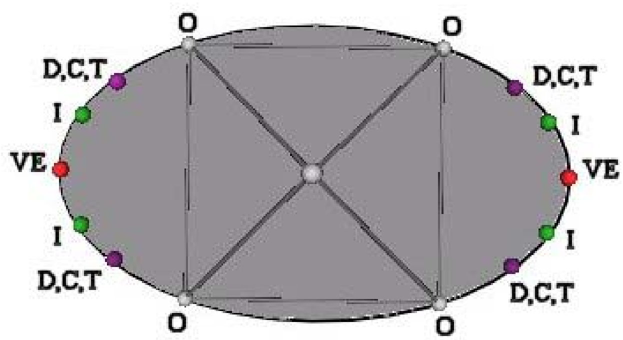

Figure #14 *Polyhedra positions of the Jitterbug motion*

Using an equation given above

$$\cos(\varphi) = \frac{\sqrt{3}}{\sqrt{2}\ EL}\ z + 1/2$$

we can calculate the angular amount φ that the Jitterbug triangle rotates from the VE position into the Icosahedron position.

Now,

$$DVF_O = \frac{1}{\sqrt{6}} EL_O \equiv \text{Distance from the center of Volume to the Face center of}$$

the Octahedron,

and with $\tau = \frac{1+\sqrt{5}}{2}$ we have

$$DVF_I = \frac{1}{2\sqrt{3}} \tau^2 \; EL_I \equiv \text{Distance from the center of Volume to the Face center}$$

of the Icosahedron.

Since the triangle face of the Jitterbug does not change scale, we have $EL_O = EL_I = EL$. We set EL = 1 for convenience. The distance z from the Octahedron face center to the Icosahedron face center is then

$$z = \frac{1}{2\sqrt{3}} \tau^2 - \frac{1}{\sqrt{6}}$$

This gives

$$\cos(\varphi) = \frac{\sqrt{3}}{\sqrt{2}} \left(\frac{1}{2\sqrt{3}} \tau^2 - \frac{1}{\sqrt{6}} \right) + 1/2$$

$$\cos(\varphi) = \frac{1}{2\sqrt{2}} \tau^2$$

which implies that $\varphi = \mathrm{acos}\left(\frac{1}{2\sqrt{2}} \tau^2 \right) \cong 22.23875609\ldots°$. That is, starting in the VE position, the 8 triangles are rotated by the amount φ (clockwise or counterclockwise) to obtain the Icosahedron position. The triangles are like gears in that if a triangle is rotated clockwise, then the 3 triangles attached to it must rotate counterclockwise.

To calculate the angular rotation of the Jitterbug triangles for the regular Dodecahedron position, we first find the radial displacement of one of the Jitterbug's triangles.

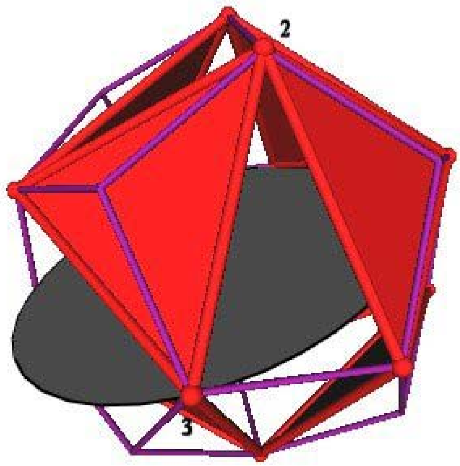

Figure #15 *Three vertices of the Jitterbug and Dodecahedron*

Figure #15 shows 3 of the Jitterbug's triangles coinciding with 3 of the Dodecahedron's vertices. It can be shown that these three vertices can be given the (x, y, z) coordinates

$V1 = (0, -\tau, \tau^3)$

$V2 = (-\tau^3, 0, \tau)$

$V3 = (-\tau, -\tau^3, 0)$

where $\tau = \frac{1+\sqrt{5}}{2}$.

Using these coordinates sets the Octahedron's edge length. The edge length of the Octahedron, calculated using the equation

$\tau^{n+1} = \tau^n + \tau^{n-1}$

is

$EL_O = \text{distance}(V1, V2) = \text{sqrt}(\tau^6 + \tau^2 + (\tau^3 - \tau)^2)$

$EL_O = 2\tau^2$

Then, using $\tau^3 = \tau^2 + \tau = \tau + 1 + \tau = 2\tau + 1$, the center of the triangle face is at

$FC = (-(\tau + 1/3), -(\tau + 1/3), (\tau + 1/3))$

which is a distance

$$DVF_{DT} = \sqrt{3}\ (\tau + 1/3)$$

from the center of volume.

With

$$DVF_O = \frac{1}{\sqrt{6}} EL_O$$

and using

$$z = DVF_{DT} - DVF_O$$

we get

$$z = \sqrt{3}\ (\tau + 1/3) - \frac{1}{\sqrt{6}} EL_O$$

Using the equation

$$\cos(\varphi) = \frac{\sqrt{3}}{\sqrt{2}\ EL}\ z + 1/2$$

and with $EL_O = EL$ we can calculate this to be

$$\cos(\varphi) = \frac{\sqrt{3}}{\sqrt{2}\ EL_O}\ \left(\sqrt{3}\ (\tau + 1/3) - \frac{1}{\sqrt{6}} EL_O \right) + 1/2$$

$$\cos(\varphi) = \frac{3\left(\tau + 1/3\right)}{2\sqrt{2}\ \tau^2}$$

so that

$$\varphi = \operatorname{acos}\left(\frac{3\left(\tau + 1/3\right)}{2\sqrt{2}\ \tau^2} \right) \cong 37.76124392\ldots^\circ.$$

This is the angular amount that the Jitterbug triangle is rotated from the VE position to the Dodecahedron position.

Sub-Octahedron Zone

As mentioned above, with physical, solid triangles, a Jitterbug's vertex does not follow the complete path of an ellipse. We now remove this constraint and let the vertices travel along the complete elliptical path. This means that the triangles of the Jitterbug may now change their size as they continue to move radially and to rotate.

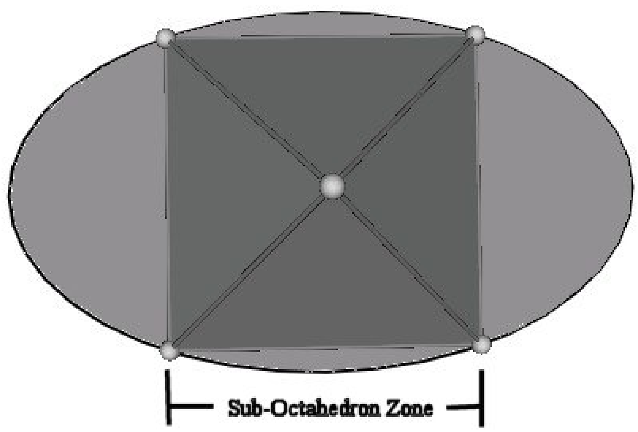

Figure #16 *The sub-Octahedron Zone of ellipse*

Beginning in the Octahedron position, the vertices are now to travel within the sub-Octahedron zone of the 6 ellipses of the Jitterbug. As shown in Figure #17, each of the Octahedron's vertices split into 2 vertices and the diametrically opposite vertices, on the same ellipse, travel in the same direction.

Note that the 3 vertices of a triangle have switched ellipses. That is, in going from the original VE position to the original Octahedron position, a vertex of a triangle follows a particular ellipse. For the triangle to continue to rotate and to remain on some elliptical path, the vertex of the triangle switches to one of the other 3 ellipse which pass through the Octahedron vertex position. The vertex, having switched, can now travel along the sub-Octahedron zone portion of a ellipse.

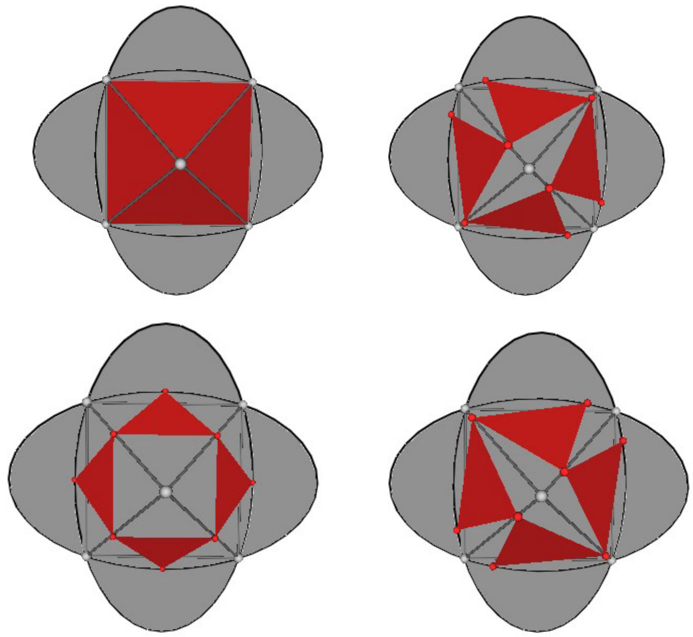

Figure #17 *Jitterbug through sub-Octahedron zone*

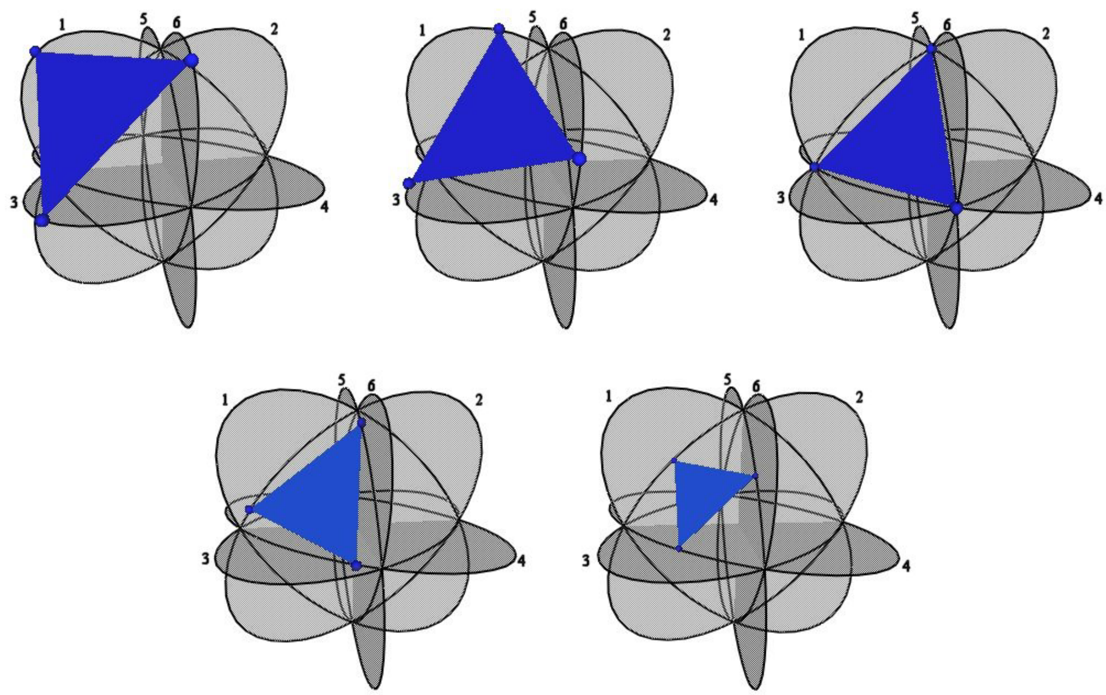

Figure #18 *Triangle vertices switch ellipses*

Figure #18 shows one triangle of the Jitterbug triangles with its 3 vertices on ellipses 1, 3, and 6. Once the triangle is in the Octahedron position, the vertices switch to follow along ellipses 5, 2, 4, respectively.

(An alternative triangle motion for traversing the sub-Octahedron zone of the ellipses will be described in the next section.)

In order to accomplish this motion, the Jitterbug triangles move radially, rotate *and change scale*. This scale change is unlike the motion of the original Jitterbug motion describe previously.

In one position it is seen that the Jitterbug forms another, smaller VE. (See Figure #17.) Being another VE configuration, we can draw another pair of smaller ellipses in each of the 3 ellipse planes. This construction of another sub-VE within the original VE by following the ellipse paths can be continued to form sub-sub-VEs, etc. and therefore sub-sub-Jitterbugs.

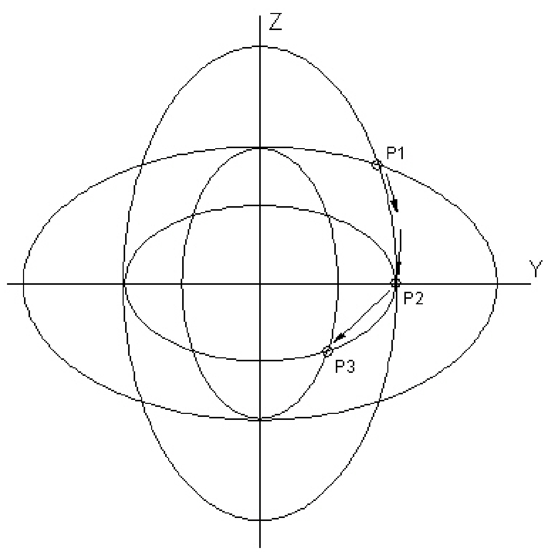

Figure #19 *First sub-Jitterbug ellipses*

As Figure #19 shows, the Octahedron vertex at P1 is moved to position P2 along the sub-Octahedron zone of the original ellipse. Again, this is not part of the normal Jitterbug motion and is accomplished by a continuous change in scale. From P2, the vertex may either continue along the original ellipse or it may smoothly switch to the smaller embedded ellipse and move to P3. P3 is a sub-Octahedron vertex position. The motion from P2 to P3 is a normal Jitterbug motion, i.e. without scaling.

As before, we can map out the various polyhedra positions of the Jitterbug motion as its vertices traverses the sub-Octahedron zone. This is shown in Figure #20 and Figure #21.

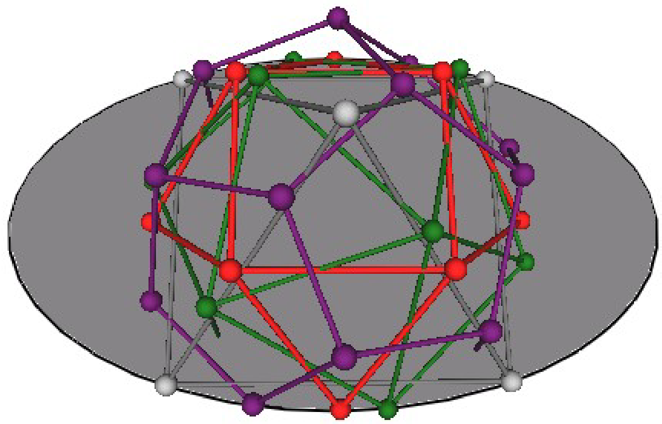

Figure #20 *One Dodecahedron, Icosahedron and VE position within sub-Octahedron Zone of ellipse*

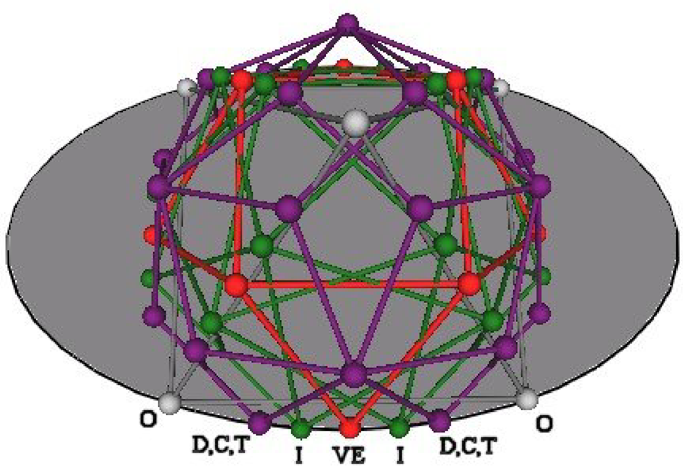

Figure #21 *Dodecahedron, Icosahedron and VE positions*

From the original, large VE, (maximum radial distance from the center of volume) a triangle will move radially inward and rotate to the original, large Octahedron position.

To then move to the sub-VE position, the triangle must reverse its radial direction (it moves radially outward) rotate (in either the same direction or opposite direction) and change scale (shrink in size.)

The equations for the radial displacement and angular rotation of the triangles are now calculated.

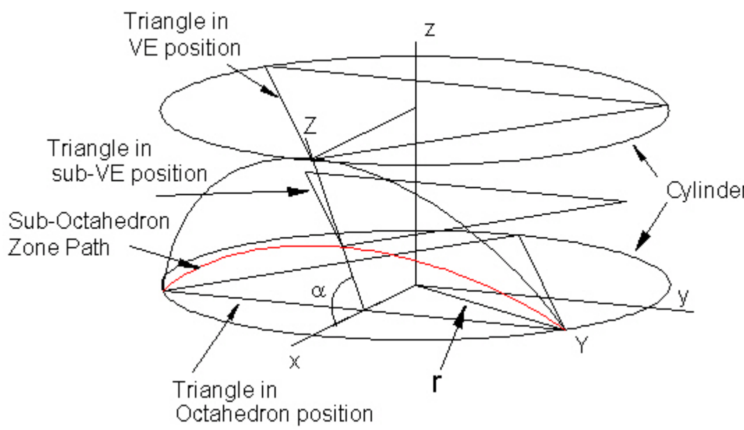

Figure #22 *Sub-Octahedron Path*

Starting with the ellipse equation (from above)

$$\frac{Y^2}{\frac{1}{3}EL^2} + \frac{Z^2}{EL^2} = 1$$

we first rotate the ellipse by 90 degrees

$$\frac{Z^2}{\frac{1}{3}EL^2} + \frac{Y^2}{EL^2} = 1$$

and solve for Z to get

$$Z = \frac{1}{\sqrt{3}} \sqrt{EL^2 - Y^2}$$

This is the equation defining the path which a vertex will travel in the YZ-plane of the original ellipse. The range of the Y variable is $(-1/2)EL \leq Y \leq (1/2)\,EL$.

When $Y = (-1/2)EL$, the triangle is a distance

$Z_0 = (1/2)\ EL$

from the coordinate origin. Then, in general, the triangle is *displaced* a distance of

$$Z_d = \frac{1}{\sqrt{3}} \sqrt{EL^2 - Y^2} - (1/2)EL$$

because the Z-axis is at an angle of $\alpha = 54.73561032\ldots°$ to the z-axis (which is the axis along which the triangle is displaced.) Recall that

$$\sin(\alpha) = \frac{\sqrt{2}}{\sqrt{3}}, \text{ and that } \cos(\alpha) = \frac{1}{\sqrt{3}}$$

so we get

$$z_d = Z_d \sin(\alpha) = \frac{\sqrt{2}}{\sqrt{3}} \left(\frac{1}{\sqrt{3}} \sqrt{EL^2 - Y^2} - (1/2)EL\right)$$

$$z_d = \frac{\sqrt{2}}{3} \sqrt{EL^2 - Y^2} - \frac{1}{\sqrt{6}} EL$$

This is the radial displacement of the triangle.

Since

$$Y = DFE_O \tan(\varphi)$$

and $DFE_O = \frac{1}{2\sqrt{3}} EL$ so that $Y^2 = (1/12)\ EL^2 \tan^2(\varphi)$ we can write this radial displacement in terms of the angular rotation of the triangle

$$z_d = \frac{\sqrt{2}}{3} \sqrt{EL^2 - (1/12)EL^2 \tan^2(\varphi)} - \frac{1}{\sqrt{6}} EL$$

with $-60° \leq \varphi \leq 60°$. Note that $\varphi = 0°$ is the small sub-VE position and that $\varphi = \pm 60°$ are the original Octahedron positions.

When the triangle is rotating from the original Octahedron position to the sub-VE position, the scale of the triangle is decreased. When the triangle further rotates from the sub-VE position to the second Octahedron position, the scale of the triangle increases back to its original size.

Projecting the path onto the xy-plane, and knowing that the distance from the Octahedron's face center to its mid-edge point DFE_O is

$$DFE_O = \frac{1}{2\sqrt{3}} EL$$

we get

$$x_d = Z_d \cos(\alpha) + DFE_O = \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{3}} \sqrt{EL^2 - Y^2} - (1/2)EL \right) + DFE_O$$

$$x_d = (1/3) \sqrt{EL^2 - Y^2} - \frac{1}{2\sqrt{3}} EL + \frac{1}{2\sqrt{3}} EL$$

$$Y^2 = (1/12)\, EL^2 \tan^2(\varphi)$$

$$x_d = (1/3) \sqrt{EL^2 - (1/12)EL^2 \tan^2(\varphi)}$$

The distance of the vertex to the z_d axis is then

$$r = \text{sqrt}(x_d^2 + Y^2)$$

Now,

$$x_d^2 = (1/9)EL^2 - (1/9)(1/12)\, EL^2 \tan^2(\varphi)$$

so

$$r^2 = (1/9)EL^2 - (1/9)(1/12)\, EL^2 \tan^2(\varphi) + (1/12)\, EL^2 \tan^2(\varphi)$$

$$r^2 = (1/9)EL^2 + (8/9)(1/12)\, EL^2 \tan^2(\varphi)$$

$$r = (1/3) \sqrt{1 + (2/3)\tan^2(\varphi)}\ EL$$

The scale factor which the triangle is reduced (as well as the associated polyhedra) as it rotates is given by

$$SF = r / DFE_O = r / \left(\frac{1}{\sqrt{3}} EL\right)$$

$$SF = \frac{1}{\sqrt{3}} \sqrt{1 + (2/3)\tan^2(\varphi)}$$

where the rotation angle $-60° \le \varphi \le 60°$.

For the sub-VE position, with $\varphi = 0°$, we get

$$SF = \frac{1}{\sqrt{3}}$$

An alternative calculation for the sub-VE position can be calculated by noting that position P2 is at the semiminor axis position of the larger ellipse and is the semimajor axis position of the smaller ellipse. Therefore, the Jitterbug in the sub-VE position is reduced by the scale factor (SF)

SF_{VE} = small ellipse semimajor axis / large semimajor axis

= large ellipse semiminor axis / large semimajor axis

$$= \frac{1}{\sqrt{3}} \text{ EL / EL}$$

$$SF_{VE} = \frac{1}{\sqrt{3}} \cong 0.577350269$$

The Scale Factor for the Dodecahedron position of the Jitterbug is now calculated.

Recall that the angle of rotation for the Dodecahedron position is

$$\varphi = \text{acos}\left(\frac{3\left(\tau + 1/3\right)}{2\sqrt{2}\ \tau^2} \right) \cong 37.76124392\ldots°.$$

so

$$\cos(\varphi) = \frac{3\left(\tau + 1/3\right)}{2\sqrt{2}\ \tau^2} = \frac{3\tau + 1}{2\sqrt{2}\ \tau^2}$$

It can be shown that

$$\sin(\varphi) = \frac{\sqrt{9\tau + 6}}{2\sqrt{2}\ \tau^2}$$

then

$$\tan(\varphi) = \sin(\varphi) / \cos(\varphi) = \frac{\sqrt{9\tau + 6}}{3\tau + 1} = \frac{\sqrt{3}}{\sqrt{5}}$$

Using this value in the Scale Factor equation, we get

$$SF_D = \frac{1}{\sqrt{3}} \sqrt{1 + (2/3)\tan^2(\varphi)} = \frac{1}{\sqrt{3}} \sqrt{1 + (2/5)}$$

So the Dodecahedron is reduced by a scale factor of

$$SF_D = \frac{\sqrt{7}}{\sqrt{15}} \cong 0.683130051\ldots.$$

Now for the Icosahedron's scale factor.

We know that the rotation angle for the Icosahedron position is

$$\varphi = \mathrm{acos}\left(\frac{1}{2\sqrt{2}}\tau^2\right) \cong 22.23875609\ldots°.$$

so that

$$\cos(\varphi) = \frac{1}{2\sqrt{2}}\tau^2$$

Then it can be shown that

$$\sin(\varphi) = \frac{\sqrt{8 - \tau^4}}{2\sqrt{2}}$$

then

$$\tan(\varphi) = \frac{\sqrt{-3\tau + 6}}{\sqrt{3\tau + 2}}$$

This gives

$$SF_I = \frac{1}{\sqrt{3}} \sqrt{1 + (2/3)\tan^2(\varphi)} = \frac{1}{\sqrt{3}} \frac{\sqrt{\tau + 6}}{\sqrt{3\tau + 2}}$$

So the Icosahedron is reduced by a scale factor of

$$SF_I = \frac{\sqrt{\tau + 6}}{\sqrt{9\tau + 6}} \cong 0.658613584\ldots.$$

Alternative Sub-Octahedron Zone Motion

There is another way for the vertices of the original sized Jitterbug to traverse the sub-Octahedron zone portion of the ellipse. With this alternative method the triangles do not change scale and they continue move radially inward. This can be accomplished by allowing the triangles to interpenetrate one another. See Figure #23. Note that the triangles' vertices are still paired. That is, the triangles are still joined together.

As Figure #23 shows, the same sequence of polyhedra (Dodecahedron, Icosahedron, VE) occurs as in the previous case.

When the vertices are in the VE position, the triangles all have their face centers at the coordinate origin (0, 0, 0).

Along the sub-Octahedron zone, the triangles rotate from 0 to 30 degrees from the original Octahedron to the sub-VE position and another 30 degrees from the sub-VE back to the original Octahedron position. From Octahedron to sub-VE position, the triangles move radially inward a distance of

$$DVF_O = \frac{1}{\sqrt{6}} \text{ EL}$$

Note that these rotations are half that of the original Jitterbug motion (the non-sub-Octahedron zone motions) but that the total radial displacement from the original Octahedron to sub-VE position is the same as the total radial displacement from the original Octahedron to the original VE position.

Robert W. Gray
rwgray@rwgrayprojects.com

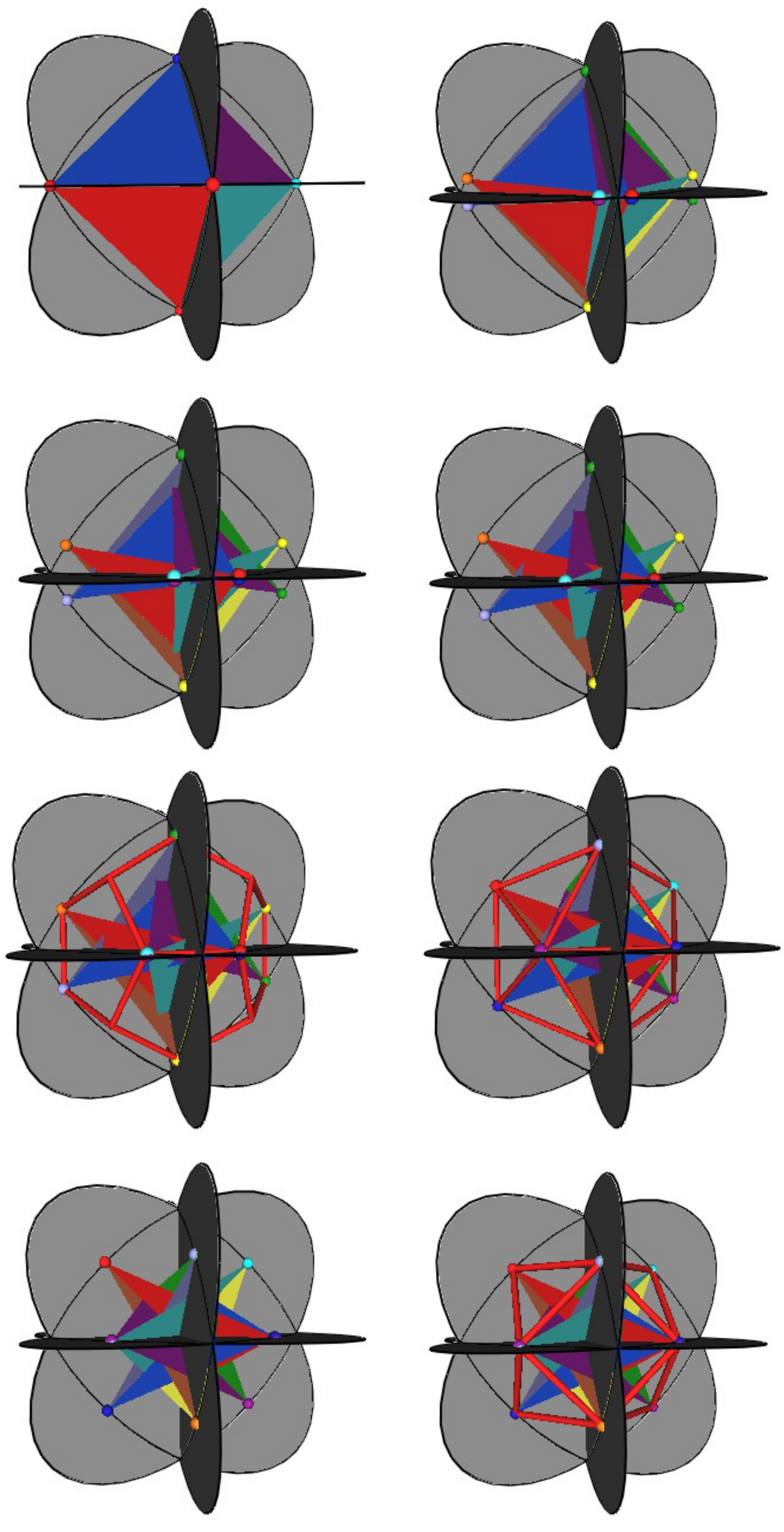

Figure #23 *Triangles are allowed to interpenetrate*

We now develop equations for the vertex motion along the sub-Octahedron zone.

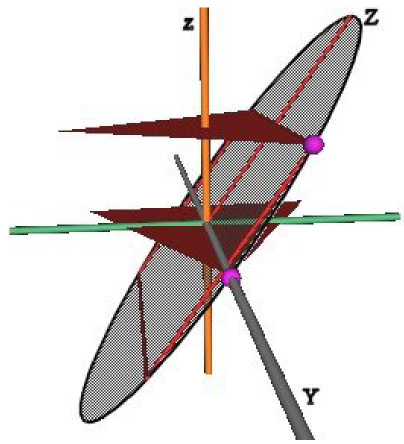

Figure 24 *Orientation of ellipse and axes*

Starting with the ellipse equation (in the YZ-plane)

$$\frac{Y^2}{\frac{1}{3}EL^2} + \frac{Z^2}{EL^2} = 1$$

we have

$$Z = \sqrt{EL^2 - 3Y^2}$$

Now,

$$z = Z \sin(\alpha)$$

and we know that $\sin(\alpha) = \frac{\sqrt{2}}{\sqrt{3}}$, and that $\cos(\alpha) = \frac{1}{\sqrt{3}}$ so

$$z = \frac{\sqrt{2}}{\sqrt{3}} \sqrt{EL^2 - 3Y^2}$$

Since

$$Y = DFV_O \cos(\varphi)$$

and $DFE_O = \frac{1}{\sqrt{3}} EL$ so that $Y^2 = (1/3)\, EL^2 \cos^2(\varphi)$

$$z = \frac{\sqrt{2}}{\sqrt{3}} \sqrt{1 - \cos^2(\varphi)} \quad EL$$

$$z = \frac{\sqrt{2}}{\sqrt{3}} \sin(\varphi) \text{ EL}$$

This gives us the displacement of the triangle's center of face along the z-axis with respect to the rotation of the triangle. Note that the angle φ ranges from

$-30° < \varphi < 30°$

with $\varphi = 0$ the sub-VE position with the triangle face center at (0,0,0).

The displacement of the triangle from the Octahedron position (at $\varphi = 30°$) as it rotates to the sub-VE position (at $\varphi = 0°$) to the second Octahedron position (at $\varphi = -30°$) is given by

$$z_d = \frac{1}{\sqrt{6}} \text{ EL} - \frac{\sqrt{2}}{\sqrt{3}} \sin(\varphi) \text{ EL}$$

Since the angular rotation range is compressed by a half, the angular amount of rotation to the various polyhedra positions are given by:

Icosahedron:

$$\varphi = (1/2) \operatorname{acos}\left(\frac{1}{2\sqrt{2}} \tau^2 \right) \cong 11.11937805\ldots°.$$

Dodecahedron:

$$\varphi = (1/2) \operatorname{acos}\left(\frac{3\left(\tau + 1/3\right)}{2\sqrt{2}\ \tau^2} \right) \cong 18.88062196\ldots°.$$

These angles are relative to the VE position at $\varphi = 0°$.

The general Scale Factor equation for the *polyhedra* is given by:

$$SF = \frac{1}{\sqrt{3}} \sqrt{1 + (2/3) \tan^2(2\,\varphi)}$$

where the rotation angle $-30° \le \varphi \le 30°$.   Note that the ***triangles do not change scale***.
The scale factors for the polyhedra are as before:

VE:

$$SF_{VE} = \frac{1}{\sqrt{3}} \cong 0.577350269$$

Icosahedron:

$$SF_{I} = \frac{\sqrt{\tau + 6}}{\sqrt{9\tau + 6}} \cong 0.658613584\ldots.$$

Dodecahedron:

$$SF_{D} = \frac{\sqrt{7}}{\sqrt{15}} \cong 0.683130051\ldots.$$

Additional Comments

The Jitterbug ellipse is such that it passes through 6 vertices of the combined odd-even FCC lattices.

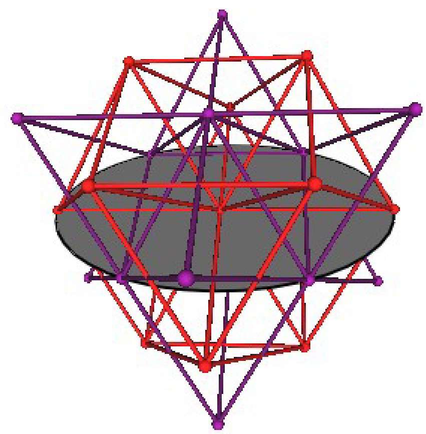

Figure 25 *Ellipse in odd-even FCC combined lattice*

In Figure #25, the red is the even (vertex centered) FCC lattice and the purple is the odd (Octahedron centered) FCC lattice.

Two Jitterbugs can not share the same triangular face *and* have their positions (location of center of volume) fixed as they go through the Jitterbug motion. If two Jitterbugs are to share the same triangle face then as the joined Jitterbugs jitterbug the positions of the Jitterbugs must move.

As Fuller points out, when in the Octahedron position, it is possible to "twist" the Jitterbug to make it collapse and lay flat. It can then be folded into a Tetrahedron.

There are many Jitterbugs, of various sizes, in the 120 Polyhedron.

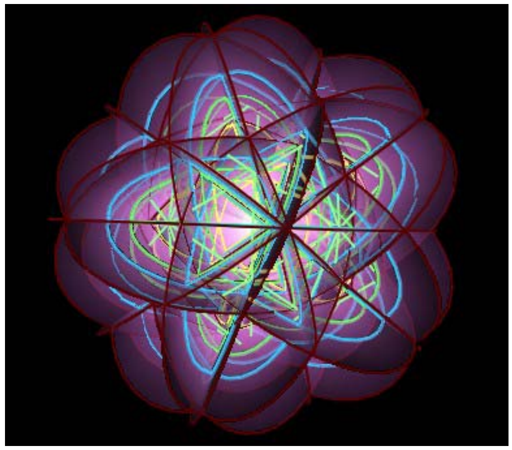

Figure 26 *Five Jitterbugs' ellipse sets in the planes of 15 Great Circles to define a 120 Polyhedron like structure*

Summary

The vertices of the Jitterbug triangles move on elliptical paths.

There are 6 ellipses per Jitterbug. These 6 ellipses define 3 planes, 2 ellipses per plane. The planes intersect each other at 90 degrees. The 2 ellipses per plane intersect each other at 90 degrees.

The equation for an ellipse is

$$\frac{Y^2}{\frac{1}{3}EL^2} + \frac{Z^2}{EL^2} = 1$$

The parametric form of the equation for the ellipse is given by

$$Y = \frac{1}{\sqrt{3}} EL \sin(\theta)$$

$$Z = EL \cos(\theta)$$

with $0° < \theta < 360°$.

The semimajor axis is $a = EL$. (EL = the edge length of the Jitterbug.)

The semiminor axes is $b = \frac{1}{\sqrt{3}} EL$.

The eccentricity of the ellipse is $e = \frac{\sqrt{2}}{\sqrt{3}} \cong 0.81649658$.

The coordinates for the 2 focus points in the (Y, Z) plane are

$$(0, \frac{\sqrt{2}}{\sqrt{3}} EL) \cong (0.0, 0.81649658\ EL) \text{ and}$$

$$(0, -\frac{\sqrt{2}}{\sqrt{3}} EL) \cong (0.0, -0.81649658\ EL)$$

Robert W. Gray
rwgray@rwgrayprojects.com

As the Jitterbug moves from an VE position to a Octahedron position, the vertices pass through first an Icosahedron position and then a regular Dodecahedron position.

The angular amount that the Jitterbug triangle is rotated (in either direction) from the VE position to the Icosahedron position:

$$\varphi = \text{acos}\left(\frac{1}{2\sqrt{2}} \tau^2 \right) \cong 22.23875609\ldots°.$$

The angular amount that the Jitterbug triangle is rotated (in either direction) from the VE position to the Dodecahedron position:

$$\varphi = \text{acos}\left(\frac{3\left(\tau + 1/3\right)}{2\sqrt{2}\ \tau^2} \right) \cong 37.76124392\ldots°.$$

As the vertices travel along the sub-Octahedron zone portion of the ellipse, the Jitterbug continuously changes scale. The Scale Factor is given by the equation

$$SF = \frac{1}{\sqrt{3}} \sqrt{1 + (2/3)\tan^2(\varphi)}$$

where the rotation angle $-60° \le \varphi \le 60°$. The angle $\varphi = 0$ corresponds to the sub-VE position and the angles $\varphi = \pm 60°$ correspond to the un-scaled, original Octahedron position.

In the sub-VE position, the Jitterbug has changed scale by

$$SF_{VE} = \frac{1}{\sqrt{3}} \cong 0.577350269$$

In the sub-Icosahedron position, the Jitterbug has change scale by

$$SF_I = \frac{\sqrt{\tau + 6}}{\sqrt{9\tau + 6}} \cong 0.658613584\ldots.$$

In the sub-Dodecahedron position, the Jitterbug has changed scale by the factor

$$SF_D = \frac{\sqrt{7}}{\sqrt{15}} \cong 0.683130051\ldots.$$

References

Fuller, R. Buckminster, Synergetics, MacMillan Publishing Company, 1982

The folowing references were ***not*** used for writing of this paper. They were discovered only after I had done my own calculations and illustrations for this paper.

Angell, I. O. and Moore, M. "Symmetrical Intersections of Cylinders." *Acta Cryst. Sect. A* **43**, 244-250, 1987.

Moore, M. "Symmetrical Intersections of Right Circular Cylinders." *Math. Gaz.* **58**, 181-185, 1974.